Further Results on Strict Lyapunov Functions for Rapidly Time-Varying Nonlinear Systems ⋆

Frédéric Mazenc [a], Michael Malisoff [b], Marcio S. de Queiroz [c]

[a] *Projet MERE INRIA-INRA, UMR Analyse des Systèmes et Biométrie INRA, 2 pl. Viala, 34060 Montpellier, France*

[b] *Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918*

[c] *Department of Mechanical Engineering, Louisiana State University, Baton Rouge, LA 70803-6413*

Abstract

We explicitly construct global strict Lyapunov functions for rapidly time-varying nonlinear control systems. The Lyapunov functions we construct are expressed in terms of oftentimes more readily available Lyapunov functions for the limiting dynamics which we assume are uniformly globally asymptotically stable. This leads to new sufficient conditions for uniform global exponential, uniform global asymptotic, and input-to-state stability of fast time-varying dynamics. We also construct strict Lyapunov functions for our systems using a strictification approach. We illustrate our results using a friction control example.

Key words: time-varying systems, input-to-state stability, Lyapunov function constructions

1 Introduction

The stabilization of nonlinear and nonautonomous control systems, and the construction of their Lyapunov functions, are challenging problems that are of significant ongoing interest; see Malisoff & Mazenc (2005) and Mazenc & Bowong (2004). One popular approach to guaranteeing stability of nonautonomous systems is the *averaging method* in which exponential stability of an appropriate *autonomous* system implies exponential stability of the original dynamics when its time variation is sufficiently fast. See Khalil (2002) for related results.

The preceding results were extended to more general rapidly time-varying systems of the form

$$\dot{x} \;=\; f(x,t,\alpha t),\;\; x\in\mathbb{R}^n,\; t\in\mathbb{R},\; \alpha>0 \qquad (1)$$

in Peuteman & Aeyels (2002), where uniform (local) exponential stability of (1) was proven for large values of the constant $\alpha > 0$, assuming a suitable limiting dynamics

$$\dot{x} \;=\; \bar{f}(x,t) \qquad (2)$$

for (1) is uniformly exponentially stable. (We specify the choice of $\bar{f}$ in our main theorem and examples below.) This generalized a result from (Hale, 1980, pp. 190-5) on a class of systems (1) satisfying certain periodicity or almost periodicity conditions. The main arguments of Peuteman & Aeyels (2002) use (partial) averaging but do not lead to explicit Lyapunov functions for (1).

In this work, we pursue a very different approach. Instead of averaging, we explicitly construct a family of Lyapunov functions for (1) that are expressed in terms of more readily available Lyapunov functions for the limiting dynamics (2), which we again assume is asymptotically stable. In addition, while Peuteman & Aeyels (2002) assumes (2) is uniformly *exponentially* stable, we allow cases where (2) is merely (uniformly) globally asymptotically stable (UGAS), in which case our conclusion is that (1) is UGAS (but not necessarily exponentially stable) when $\alpha > 0$ is sufficiently large. While global exponential and global asymptotic stabilities are equivalent for *autonomous* systems under a coordinate change in certain dimensions, the coordinate changes are not explicit and so do not lend themselves to explicit Lyapunov function constructions; see Grüne *et al.* (1999). The importance of the problem we consider lies in the ubiquity of rapidly time-varying systems in practical applications (e.g., suspended pendulums subject to vertical vibrations of small amplitude and high frequency and Raleigh's and Duffing's equations from (Khalil, 2002,

⋆ Corresponding Author: Frédéric Mazenc. The second author was supported by NSF/DMS Grant 0424011.

Email addresses: Frederic.Mazenc@ensam.inra.fr (Frédéric Mazenc), malisoff@lsu.edu (Michael Malisoff), dequeiroz@me.lsu.edu (Marcio S. de Queiroz).

Chatper 8), as well as systems arising in identification as discussed in Peuteman & Aeyels (2002) and below) and the essentialness of Lyapunov functions in robustness analysis and controller design for these applications.

In particular, we show that assumptions similar to those of (Peuteman & Aeyels, 2002, Theorem 3) imply that (1) is uniformly globally (rather than merely locally) exponentially stable; our Lyapunov function constructions are new even in this particular case and our results are complementary to those of Peuteman & Aeyels (2002). The Lyapunov functions we construct are also input-to-state stable (ISS) or integral ISS Lyapunov functions for the rapidly time-varying control system

$$\dot{x} \;=\; f(x,t,\alpha t) + g(x,t,\alpha t)u \qquad (3)$$

under appropriate conditions on f and g; see Remark 4.

In Section 2, we provide the relevant definitions and lemmas. In Section 3.1, we present our main sufficient conditions for uniform global asymptotic and exponential stability of (1), and for the stability of (3), in terms of limiting dynamics (2). This theorem leads to explicit constructions of Lyapunov functions for (1) and (3) in terms of Lyapunov functions for (2), in Theorem 5. In Section 3.2, we provide an alternative Lyapunov function construction theorem for (1) not involving any limiting dynamics. We prove our main results in Sections 4 and 6. We illustrate our theorems in Sections 6 and 7 using a friction control model and other examples. We close in Section 8 by summarizing our findings.

2 Assumptions, definitions, and lemmas

We study (1) (which includes dynamics (2) with no α dependence, as special cases) in which we always assume f is continuous in time $t \in \mathbb{R} := (-\infty, +\infty)$, continuously differentiable (C^1) in $x \in \mathbb{R}^n$, null at $x = 0$ meaning

$$f(0,t,\alpha t) = \bar{f}(0,t) = 0 \quad \forall t \in \mathbb{R},\ \alpha > 0 \qquad (4)$$

and *forward complete*, i.e., for each $\alpha > 0$, $x_o \in \mathbb{R}^n$, and $t_o \in \mathbb{R}_{\geq 0} := [0,\infty)$ there exists a unique trajectory $[t_0,\infty) \ni t \mapsto \phi(t;t_o,x_o)$ for (1) (depending in general on the constant $\alpha > 0$) that satisfies $x(t_o) = x_o$. We set $\mathbb{N} = \{1,2,3,\ldots\}$ and let $\mathbb{Z}$ denote the set of all integers. We say $N : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{M}$ and write $N \in \mathcal{M}$ provided

$$\lim_{\eta \to +\infty} \eta N(\eta) = 0. \qquad (5)$$

A continuous function $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *positive definite* provided it is zero only at zero. A positive definite function δ is *of class* $\mathcal{K}$ (written $\delta \in \mathcal{K}$) provided it is strictly increasing; if in addition δ is unbounded, then we say that δ is *of class* $\mathcal{K}_\infty$ and write $\delta \in \mathcal{K}_\infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{KL}$ (written $\beta \in \mathcal{KL}$) provided (a) $\beta(\cdot,t) \in \mathcal{K}_\infty$ for all $t \geq 0$, (b) $\beta(s,\cdot)$ is nonincreasing for all $s \geq 0$, and (c) for each $s \geq 0$, $\beta(s,t) \to 0$ as $t \to +\infty$. A positive definite function δ is called $o(s)$ provided $\delta(s)/s \to 0$ as $s \to +\infty$.

We next define our stability properties for (2). The same definitions apply for (1) for any choice of the constant $\alpha > 0$. We call (2) *uniformly globally asymptotically stable (UGAS)* provided there exists $\beta \in \mathcal{KL}$ such that

$$|\phi(t;t_o,x_o)| \;\leq\; \beta(|x_o|, t-t_o)\ \forall t \geq t_o \geq 0,\ x_o \in \mathbb{R}^n \qquad (6)$$

where $|\cdot|$ is the usual Euclidean norm and ϕ is the flow map for (2). We call (2) *uniformly globally exponentially stable (UGES)* provided there exist constants $D > 1$ and $\lambda > 0$ such that (6) is satisfied with the choice

$$\beta(s,t) \;=\; Dse^{-\lambda t}. \qquad (7)$$

The converse Lyapunov function theorem implies (2) is UGAS if and only if it has a *(strict) Lyapunov function*, i.e., a C^1 $V : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that admits $\delta_1, \delta_2 \in \mathcal{K}_\infty$ and $\delta_3 \in \mathcal{K}$ such that for all $t \in \mathbb{R}_{\geq 0}$ and $\xi \in \mathbb{R}^n$, we have both (L1) $\delta_1(|\xi|) \leq V(\xi,t) \leq \delta_2(|\xi|)$ and (L2) $V_t(\xi,t) + V_\xi(\xi,t)\,\bar{f}(\xi,t) \leq -\delta_3(|\xi|)$, where the subscripts on V denote partial gradients; see Bacciotti & Rosier (2005). When (2) is UGES, the proof of (Khalil, 2002, Theorem 4.14) shows:

Lemma 1 *Assume (2) satisfies the UGES condition (6)-(7) for some constants $D > 1$ and $\lambda > 0$ and that there exists $K > \lambda$ such that $|(\partial \bar{f}/\partial \xi)(\xi,t)| \leq K$ for all $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}_{\geq 0}$. Then (2) admits a Lyapunov function V and constants $c_1, c_2, c_3 > 0$ such that*

$$\begin{array}{l} c_1|\xi|^2 \;\leq\; V(\xi,t) \;\leq\; c_2|\xi|^2\ ,\quad |V_\xi(\xi,t)| \;\leq\; c_3|\xi|\ , \\ V_t(\xi,t) + V_\xi(\xi,t)\bar{f}(\xi,t) \;\leq\; -|\xi|^2. \end{array} \qquad (8)$$

hold for all $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{R}^n$.

Motivated by Lemma 1, we find it convenient to use the following compatibility condition for UGAS systems (2):

Definition 1 *Given $\delta \in \mathcal{K}$, the dynamics (2) is said to be* δ-compatible *provided it admits a Lyapunov function $V \in C^1$ and two constants $\bar{c} \in (0,1)$, $\bar{\bar{c}} > 0$ such that:*

- P_1 $\quad V_t(\xi,t) + V_\xi(\xi,t)\,\bar{f}(\xi,t) \;\leq\; -\bar{c}\,\delta^2(|\xi|) \quad \forall \xi, t.$
- P_2 $\quad |V_\xi(\xi,t)| \;\leq\; \delta(|\xi|)$ and $|\bar{f}(\xi,t)| \;\leq\; \delta(|\xi|/2) \quad \forall \xi, t.$
- P_3 $\quad \delta(s) \;\leq\; \bar{\bar{c}}\,s \quad \forall s \geq 0.$

Remark 2 Note the *asymmetry* in the bounds on $|V_\xi|$ and $|\bar{f}|$ in P_2. If (2) satisfies the assumptions of Lemma 1, then it is δ-compatible with $\delta(s) = (c_3 + 2K)s$. However, by varying δ (including cases where δ is bounded), one finds a rich class of non-UGES δ-compatible dynamics as well; see e.g. Section 6.1 below.

We also consider the nonautonomous *control system*

$$\dot{x} = F(x, t, u) \tag{9}$$

which we always assume is continuous in all variables and C^1 in x with $F(0, t, 0) \equiv 0$, and whose solution for a given control function $\mathbf{u} \in \mathcal{U}$(:=all measurable locally essentially bounded functions $[0, \infty) \to \mathbb{R}^m$) and given initial condition $x(t_o) = x_o$ we denote by $t \mapsto \phi(t; t_o, x_o, \mathbf{u})$. We always assume (9) is *forward complete*, i.e., all trajectories $\phi(\cdot; t_o, x_o, \mathbf{u})$ so defined have domain $[t_o, +\infty)$. We next recall the input-to-state stable (ISS) and integral input-to-state stable (iISS) properties from Sontag (1989) and Sontag (1998). Let $|\mathbf{u}|_I$ denote the essential supremum of $\mathbf{u} \in \mathcal{U}$ restricted to any interval $I \subseteq \mathbb{R}_{\geq 0}$.

Definition 2 *(a) We say that (9) is* ISS *provided there exist $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ for which*

$$|\phi(t; t_o, x_o, \mathbf{u})| \;\leq\; \beta(|x_o|, t - t_o) + \gamma\left(|\mathbf{u}|_{[t_o, t+t_o]}\right) \tag{10}$$

holds when $t \geq t_o \geq 0$, $x_o \in \mathbb{R}^n$, and $\mathbf{u} \in \mathcal{U}$. If in addition β has the form (7), then we say that (9) is input-to-state exponentially stable (ISES). *(b) We say that (9) is* iISS *provided there exist $\mu, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that*

$$\mu(|\phi(t; t_o, x_o, \mathbf{u})|) \;\leq\; \beta(|x_o|, t - t_o) + \int_{t_o}^{t_o+t} \gamma(|\mathbf{u}(s)|)\, \mathrm{d}s$$

holds when $t \geq t_o \geq 0$, $x_o \in \mathbb{R}^n$, and $\mathbf{u} \in \mathcal{U}$.

A function $V : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called *uniformly positive definite* provided $s \mapsto \inf\{V(x, t) : t \geq 0, |x| = s\}$ is positive definite in which case we write $V \in$ UPD. We call $V \in$ UPD *uniformly proper and positive definite* provided there exist $\delta_1, \delta_2 \in \mathcal{K}_\infty$ such that condition (L1) above is satisfied in which case we write $V \in$ UPPD. The following Lyapunov function notions agree with the usual ISS and iISS Lyapunov function definitions when (9) is autonomous, because functions $\chi \in \mathcal{K}_\infty$ are invertible. Note that ν in (11) *need not* be of class $\mathcal{K}$.

Definition 3 *Let $V \in C^1 \cap$*UPPD. *(a) We call V an* ISS Lyapunov function *for (9) provided there exist $\chi, \delta_3 \in \mathcal{K}_\infty$ such that for all $t \in \mathbb{R}_{\geq 0}$, $\xi \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$: $[|u| \leq \chi(|\xi|) \Rightarrow V_t(\xi, t) + V_\xi(\xi, t)\, F(\xi, t, u) \leq -\delta_3(|\xi|)]$. (b) We call V an* iISS Lyapunov function *for (9) provided there exist $\Delta \in \mathcal{K}_\infty$ and a positive definite function $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$V_t(\xi, t) + V_\xi(\xi, t)\, F(\xi, t, u) \;\leq\; -\nu(|\xi|) + \Delta(|u|) \tag{11}$$

holds for all $t \in \mathbb{R}_{\geq 0}$, $\xi \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$.

Since (9) has an ISS Lyapunov function when it is ISS (by the arguments of Sontag & Wang (1995)), the proof of (Angeli *et al.*, 2000, Theorem 1) shows that if (9) is ISS, then it is also iISS, but not conversely, since e.g. $\dot{x} = -\arctan(x) + u$ is iISS but not ISS. The next lemma follows from the arguments used in Angeli *et al.* (2000), Edwards *et al.* (2000), and Sontag (1989).

Lemma 3 *If (9) admits an ISS (resp., iISS) Lyapunov function, then it is ISS (resp., iISS).*

3 Statements and discussions of main results

3.1 Main theorem and Lyapunov function construction

We show that the main hypotheses of (Peuteman & Aeyels, 2002, Theorem 3) ensure that (1) is UGES. In fact, we show the conditions imply

$$\dot{x} \;=\; f(x, t, \alpha t) + u, \quad x \in \mathbb{R}^n,\; u \in \mathbb{R}^n \tag{12}$$

is ISS when $\alpha > 0$ is sufficiently large; see Remark 4 below for results on the more general systems (3). Our main assumption will be: There exist $\delta \in \mathcal{K}$, a δ-compatible dynamics (2), and $N \in \mathcal{M}$ (cf. (5) above) such that for all $x \in \mathbb{R}^n$, all $r \in \mathbb{R}$ and sufficiently large $\eta > 0$,

$$\left| \int_{r-\frac{1}{\eta}}^{r+\frac{1}{\eta}} \left\{ f(x, l, \eta^2 l) - \bar{f}(x, l) \right\} \mathrm{d}l \right| \leq \delta(|x|/2) N(\eta) \tag{13}$$

of which (Peuteman & Aeyels, 2002, Property 2) is the special case where $\delta(s) = 2s$. Two more advantages of our result are (a) it applies to cases where (2) is UGAS but not necessarily UGES (cf. Section 6.1 below) and (b) its proof leads to explicit Lyapunov functions for (3) (cf. Theorem 5 below). See also Theorem 6 below for cases where $\partial f / \partial x$ is not necessarily globally bounded. Recall the definition of compatibility (in Definition 1) and the requirement $N \in \mathcal{M}$ in (5).

Theorem 4 *Consider a system (1). Assume there exist $\delta \in \mathcal{K}$, a δ-compatible UGAS system (2), two constants $\eta_o > 0$ and $K > 1$, and $N \in \mathcal{M}$ such that (13) holds whenever $\eta \geq \eta_o$, $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$ and such that:*

$$\begin{aligned} &\left| \tfrac{\partial \bar{f}}{\partial x}(x, t) \right| \;\leq\; K \;\;,\;\; \left| \tfrac{\partial f}{\partial x}(x, t, \alpha t) \right| \;\leq\; K \;\;, \;\; \text{and} \\ &|f(x, t, \alpha t)| \;\leq\; \delta(|x|/2) \quad \forall t \in \mathbb{R},\, x \in \mathbb{R}^n,\, \alpha > 0. \end{aligned} \tag{14}$$

Then there is a constant $\underline{\alpha} > 0$ such that for all constants $\alpha \geq \underline{\alpha}$, (1) is UGAS and (12) is iISS. If in addition $\delta \in \mathcal{K}_\infty$, then (12) is ISS for all constants $\alpha \geq \underline{\alpha}$. In the special case where (2) is UGES, (1) is UGES for all constants $\alpha \geq \underline{\alpha}$ and (12) is ISES for all constants $\alpha \geq \underline{\alpha}$.

By (4), the condition $|f(x, t, \alpha t)| \leq \delta(|x|/2)$ in (14) is redundant when δ has the form $\delta(s) = \bar{r}s$ for a constant $\bar{r} > 0$, since $\bar{r}$ can always be enlarged. Our proof of Theorem 4 in Section 4 below will also show:

Theorem 5 *Let the hypotheses of Theorem 4 hold for some $\delta \in \mathcal{K}$, and $V \in C^1$ satisfy the requirements of Definition 1. Then there exists a constant $\underline{\alpha} > 0$ such that for all constants $\alpha > \underline{\alpha}$,*

$$V^{[\alpha]}(\xi, t) := V\left(\xi - \tfrac{\sqrt{\alpha}}{2}\int_{t-\frac{2}{\sqrt{\alpha}}}^{t}\int_s^t \{f(\xi, l, \alpha l) - \bar{f}(\xi, l)\}\,\mathrm{d}l\,\mathrm{d}s, t\right)$$

is a Lyapunov function for (1) and an iISS Lyapunov function for (12). If also $\delta \in \mathcal{K}_\infty$, then $V^{[\alpha]}$ is also an ISS Lyapunov function for (12) for all constants $\alpha > \underline{\alpha}$.

3.2 Alternative result

The proof of Theorem 4 constructs strict Lyapunov functions for (1) in terms of strict Lyapunov functions for the limiting dynamics (2). It is natural to inquire whether one can instead construct strict Lyapunov functions for (1) by strictifying nonstrict Lyapunov functions for (1); see Malisoff & Mazenc (2005) where the strictification approach was applied to nonautonomous systems that are not rapidly time-varying. In this section, we extend this approach to cover (1). Our strictification result has the advantages in certain situations that (a) it does not require any knowledge of limiting dynamics, (b) it allows the derivative of the nonstrict Lyapunov function to be zero or even positive at some points, and (c) it does not require (1) to be globally Lipschitz in the state. In the rest of the section, we focus on systems with no controls, but the extension to the control system (3) for appropriate g can be done by similar arguments. We assume:

H. There exist $V \in C^1 \cap \mathrm{UPPD}$, $W \in \mathrm{UPD}$, a C^1 function $\Theta : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, a bounded continuous function $p : \mathbb{R} \to \mathbb{R}$, and constants $c, T > 0$ such that for all $x \in \mathbb{R}^n$, $t \geq 0$, $\alpha > 0$, and $k \in \mathbb{Z}$, we have:
  - H1. $V_t + V_x f(x, t, \alpha t) \leq -W(x, t) + p(\alpha t)\Theta(x, t)$
  - H2. $\int_{kT}^{(k+1)T} p(r)\mathrm{d}r = 0$
  - H3. $V \geq c|\Theta|$, $W \geq c \max\{|\Theta|, |\Theta_t + \Theta_x f(x, t, \alpha t)|\}$

Here and in the sequel, we omit the argument (x, t) of V, W, Θ, and the partial gradients of V and Θ whenever this would not lead to confusion. In Section 5, we prove:

Theorem 6 *If Assumption H. holds, then there exists a constant $\underline{\alpha} > 0$ such that for all constants $\alpha \geq \underline{\alpha}$,*

$$U^{[\alpha]}(x, t) = V(x, t) - \left(\int_{t-1}^{t}\left(\int_s^t p(\alpha l)dl\right) ds\right)\Theta(x, t) \tag{15}$$

is a Lyapunov function for (1). In particular, (1) is UGAS for all constants $\alpha \geq \underline{\alpha}$.

4 Proof of Theorems 4 and 5 and remarks

To make our arguments easy to follow, we first outline our method for proving these theorems. First, we give the proof of Theorem 4 for the special case where (2) is UGAS and $\delta \in \mathcal{K}_\infty$ which includes the proof of Theorem 5 for the $\delta \in \mathcal{K}_\infty$ case. Then we indicate the changes required if $\delta \in \mathcal{K}$ is bounded. Finally, we specialize to the special case where $\delta(s) = \bar{r}s$ for some constant $\bar{r} > 0$ which will prove the ISES assertion of Theorem 4.

We first assume that (2) is UGAS and $\delta \in \mathcal{K}_\infty$, and we prove the ISS property for (12) for large constants $\alpha > 0$. In what follows, we assume all inequalities and equalities hold wherever they make sense, unless otherwise indicated. Let η_o be as in the statement of the theorem, and fix $\alpha = \eta^2$ with $\eta \geq \eta_o$, $\mathbf{u} \in \mathcal{U}$, and a trajectory $x(t)$ for (12) and $\mathbf{u}$, with arbitrary initial condition. Set

$$z(t) = x(t) + R_\alpha(x(t), t), \tag{16}$$

where

$$R_\alpha(x, t) = -\frac{\eta}{2}\int_{t-2/\eta}^{t}\int_s^t \left\{f(x, l, \eta^2 l) - \bar{f}(x, l)\right\}\mathrm{d}l\,\mathrm{d}s.$$

This is well-defined since we are assuming our dynamics are forward complete. Set $p(t, l) = f(x(t), l, \eta^2 l) - \bar{f}(x(t), l)$. With p so defined, one easily checks (as was done e.g. in Malisoff & Mazenc (2005)) that for any $\tau > 0$ and $t \in \mathbb{R}$,

$$\begin{aligned}&\frac{d}{dt}\int_{t-\tau}^{t}\int_s^t p(t, l)\;\mathrm{d}l\,\mathrm{d}s = \\ &\tau p(t, t) - \textstyle\int_{t-\tau}^{t} p(t, l)\;\mathrm{d}l + \displaystyle\int_{t-\tau}^{t}\int_s^t \frac{\partial p}{\partial t}(t, l)\;\mathrm{d}l\,\mathrm{d}s\end{aligned} \tag{17}$$

and

$$\left|\int_{t-\tau}^{t}\int_s^t p(t, l)\;\mathrm{d}l\,\mathrm{d}s\right| \leq \frac{\tau^2}{2}\max_{t-\tau \leq l \leq t}|p(t, l)|. \tag{18}$$

Taking $\tau = 2/\eta$, (17) multiplied through by $-\eta/2$, $f(x(t), t, \eta^2 t) - p(t, t) \equiv \bar{f}(x(t), t)$, and (16) give

$$\begin{aligned}\dot{z}(t) &= \bar{f}(z(t), t) + \left(\bar{f}(x(t), t) - \bar{f}(z(t), t)\right) + \mathbf{u}(t) \\ &+ \tfrac{\eta}{2}\textstyle\int_{t-\frac{2}{\eta}}^{t} p(t, l)\,\mathrm{d}l \\ &- \tfrac{\eta}{2}\left\{\textstyle\int_{t-\frac{2}{\eta}}^{t}\int_s^t \left(\frac{\partial f}{\partial x}(x(t), l, \eta^2 l) - \frac{\partial \bar{f}}{\partial x}(x(t), l)\right)\mathrm{d}l\,\mathrm{d}s\right\} \\ &\times \left(f(x(t), t, \eta^2 t) + \mathbf{u}(t)\right).\end{aligned} \tag{19}$$

Let V, δ_1, and δ_2 satisfy the requirements P_1-P_3 from our compatibility condition (in Definition 1) and (L1). By P_1 with $\xi = z(t)$ and the preceding formula for $\dot{z}(t)$, the derivative of $V(z, t)$ along the time-varying map $z(t)$

defined in (16) (which we denote simply by $\dot V$ in the sequel) satisfies

$$\begin{array}{l}\dot V \ \le\ -\bar c\,\delta^2(|z(t)|) + V_\xi(z(t),t)\left(\bar f(x(t),t) - \bar f(z(t),t)\right)\\ +\frac{\eta}{2}V_\xi(z(t),t)\int_{t-2/\eta}^t p(t,l)\,\mathrm{d}l\\ -\frac{\eta}{2}V_\xi(z(t),t)\\ \times\left[\int_{t-2/\eta}^t\int_s^t\left(\frac{\partial f}{\partial x}(x(t),l,\eta^2 l) - \frac{\partial \bar f}{\partial x}(x(t),l)\right)\mathrm{d}l\,\mathrm{d}s\right]\\ \times\left(f(x(t),t,\eta^2 t)+\mathbf{u}(t)\right) + V_\xi(z(t),t)\,\mathbf{u}(t).\end{array}$$

We deduce from (13), (14), (16), and P_2 that

$$\begin{array}{rl}\dot V \ \le & -\bar c\,\delta^2(|z(t)|) + K\delta(|z(t)|)|x(t)-z(t)|\\ & +\frac{\eta}{2}\,\delta(|z(t)|)\left|\int_{t-2/\eta}^t p(t,l)\,\mathrm{d}l\right| + \delta(|z(t)|)\,|\mathbf{u}(t)|\\ & +\frac{\eta}{2}\,\left(|f(x(t),t,\eta^2 t)| + |\mathbf{u}(t)|\right)\delta(|z(t)|)\\ & \times\int_{t-2/\eta}^t\int_s^t 2K\,\mathrm{d}l\,\mathrm{d}s\\ \le & -\bar c\,\delta^2(|z(t)|) + K\delta(|z(t)|)|R_\alpha(x(t),t)|\\ & +\frac{\eta}{2}\delta(|z(t)|)N(\eta)\,\delta(|x(t)|/2) + \delta(|z(t)|)\,|\mathbf{u}(t)|\\ & +\frac{2}{\eta}K\,\delta(|z(t)|)\{\delta(|x(t)|/2) + |\mathbf{u}(t)|\}.\end{array}$$

Moreover, (14), the definition of p, and P_2 give

$$\begin{array}{rl}|R_\alpha(x(t),t)| & \le\ \frac{\eta}{2}\int_{t-2/\eta}^t\int_s^t|p(t,l)|\,\mathrm{d}l\,\mathrm{d}s\\ & \le\ \frac{2}{\eta}\delta(|x(t)|/2)\,.\end{array}\qquad(20)$$

Combining these inequalities and grouping terms gives

$$\begin{array}{rl}\dot V \ \le & -\bar c\,\delta^2(|z(t)|) + \delta(|z(t)|)\,|\mathbf{u}(t)|\\ & +\delta(|z(t)|)\left(\frac{4}{\eta}K + \frac{\eta}{2}N(\eta)\right)\{\delta(|x(t)|/2) + |\mathbf{u}(t)|\}\,.\end{array}$$

On the other hand, (16), (20), and P_3 give

$$|z(t)|\ \ge\ |x(t)| - \frac{\bar{\bar c}}{\eta}|x(t)|\ \ge\ \frac{1}{2}|x(t)| \qquad (21)$$

when $\eta \ge \max\{2\bar{\bar c}, \eta_o\}$. Since $\delta\in\mathcal{K}$, this gives

$$\begin{array}{rl}\dot V \ \le & \left(-\bar c + \frac{4}{\eta}K + \frac{\eta}{2}N(\eta)\right)\delta^2(|z(t)|)\\ & +\left(\frac{4}{\eta}K + \frac{\eta}{2}N(\eta)+1\right)\delta(|z(t)|)|\mathbf{u}(t)|.\end{array}\qquad(22)$$

Setting $\chi(s) = \frac{\bar c}{4}\delta(s/2)$, it follows from (21)-(22) that

$$\begin{array}{l}|\mathbf{u}|_\infty\ \le\ \chi(|x(t)|)\quad\Rightarrow\quad |\mathbf{u}|_\infty\ \le\ \chi(2|z(t)|)\\ \Rightarrow\quad \dot V \le \left(-\frac{3\bar c}{4} + \frac{8}{\eta}K + \eta N(\eta)\right)\delta^2(|z(t)|).\end{array}\qquad(23)$$

Setting $V^{[\alpha]}(x,t) := V(x + R_\alpha(x,t),t)$, we see the derivative $\dot V = V_t(z,t) + V_\xi(z,t)\dot z$ of $V(z,t)$ along (12) satisfies

$$\dot V = V_t^{[\alpha]}(x,t) + V_x^{[\alpha]}(x,t)\,\{f(x,t,\alpha t) + \mathbf{u}(t)\}. \qquad (24)$$

We deduce from (5), (21), and (23) that when the constant α (and so also η) is sufficiently large,

$$\begin{array}{l}|u| \le \chi(|x|)\quad\Rightarrow\\ V_t^{[\alpha]}(x,t) + V_x^{[\alpha]}(x,t)\,[f(x,t,\alpha t) + u] \le -\frac{\bar c}{2}\delta^2(|x|/2)\end{array}$$

and $\delta_1(|x|/2)\ \le\ V^{[\alpha]}(x,t)\ \le\ \delta_2(|x| + 2\delta(|x|/2)/\eta)$ by (20). It follows that $V^{[\alpha]}$ is an ISS Lyapunov function for (12), so (12) is ISS for large α, by Lemma 3, as claimed. The UGAS conclusion is the special case where $\mathbf{u}\equiv 0$. Recalling that the ISS property implies the iISS property, our iISS assertion follows if $\delta\in\mathcal{K}_\infty$. To show that (12) is iISS when $\delta\in\mathcal{K}$ is bounded, we instead follow the preceding argument up through (22) (which did not use the unboundedness of $\delta\in\mathcal{K}_\infty$) and bound the coefficient of $|\mathbf{u}|$ in (22) to show that $V^{[\alpha]}$ is an iISS Lyapunov function for (12) for sufficiently large α, which again implies that (12) is iISS for large α, by Lemma 3.

We turn next to the special case where (2) is UGES. Let V satisfy the requirements of Lemma 1 above for (2), and let $x(t)$ be any trajectory for (12) for any control $\mathbf{u}\in\mathcal{U}$. Define $z(t)$ by (16). Arguing exactly as before except with this new choice of V shows $|R_\alpha(x(t),t)| \le 2K|x(t)|/\eta$ (by taking $\delta(s) = Ks$ in (20)) and that (24) satisfies (by P_3 and (13))

$$\begin{array}{rl}\dot V \ \le & -|z(t)|^2 + c_3|z(t)||\mathbf{u}(t)|\\ & +c_3\,|z(t)|\left(\frac{4}{\eta}K^2 + \frac{\bar{\bar c}\eta}{2}N(\eta)\right)\{|x(t)| + |\mathbf{u}(t)|\}\\ \le & \left(-1 + \frac{8}{\eta}c_3K^2 + \bar{\bar c}\,\eta c_3 N(\eta)\right)|z(t)|^2\\ & +c_3|z(t)|\left(\frac{4}{\eta}K^2 + \frac{\bar{\bar c}\eta}{2}N(\eta) + 1\right)|\mathbf{u}(t)|,\end{array}$$

since $|x(t)| \le 2|z(t)|$ for large η as before. If we now define $\tilde\chi\in\mathcal{K}_\infty$ by $\tilde\chi(s) = s/\{8(1+c_3)\}$, then we deduce as in the UGAS case that if η is large enough, and if $|\mathbf{u}|_\infty \le \tilde\chi(|x(t)|)$ for all t, then we also have $|\mathbf{u}|_\infty \le \tilde\chi(2|z(t)|)$ for all t and $\dot V \le -|z(t)|^2/2 \le -V(z(t),t)/(2c_2)$. This gives $V(z(t),t)\ \le\ V(z(0),0)e^{-t/(2c_2)}$, so

$$\frac{c_1}{4}|x(t)|^2\ \le\ c_1|z(t)|^2\ \le\ V(z(t),t)\ \le\ c_2|z(0)|^2 e^{-\frac{1}{2c_2}t},$$

so our estimate on $|R_\alpha(x(t),t)|$ and the form of $z(t)$ give

$$|x(t)|\ \le\ \sqrt{\tfrac{4c_2}{c_1}}\left(1 + \tfrac{2K}{\eta}\right)|x(0)|e^{-\frac{1}{4c_2}t}. \qquad (25)$$

We conclude as before that if (2) is UGES, then, when the constant $\alpha > 0$ is large enough, (1) is also UGES and (e.g., by the proof of (Sontag & Wang, 1995, Lemma 2.14)) (12) is ISES, which proves our theorem.

Remark 4 The method we used in the proof of Theorem 4 can be used to prove the ISS property for (3) under appropriate growth assumptions on the matrix-valued function $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times m}$. Clearly, some growth condition on g is needed and linear growth of g is not enough, since $\dot{x} = -x + xu$ is not ISS. One way to extend our theorem to (3) is to add the hypothesis that there is a constant $c_o > 1$ such that for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $\alpha > 0$, $||g(x, t, \alpha t)|| \le c_o + (\delta(|x|/2))^{1/2}$, where $||\cdot||$ is the 2-norm on $\mathbb{R}^{n\times m}$ and $\delta \in \mathcal{K}_\infty$ satisfies P_1-P_3 for some Lyapunov function V for (2). Applying the first part of the proof of Theorem 4 except with the new $\mathcal{K}_\infty$ function

$$\chi(s) = \frac{\bar{c}\,\delta(s/2)}{4\{c_o + \sqrt{\delta(s/2)}\}}, \tag{26}$$

we then conclude as before that (3) is ISS for sufficiently large $\alpha > 0$. If instead $\delta \in \mathcal{K}$ is bounded, then (3) is iISS when α is sufficiently large, by our earlier argument.

Remark 5 The decay requirement (5) on $N \in \mathcal{M}$ from Theorem 4 can be relaxed, as follows. We assume the flow map ϕ of (2) satisfies the UGES conditions (6)-(7) for some $D > 1$ and $\lambda \in (0, K)$, where K satisfies (14), and we let V be as in Lemma 1. We can choose the constant c_3 in (8) to be

$$c_3 = \frac{4D(\Theta - 1)}{(K - \lambda)}, \quad \text{where } \Theta = (\sqrt{2}D)^{K/\lambda - 1}, \tag{27}$$

by the proof of Lemma 1. It follows from our argument above that Theorem 4 remains true for cases where (2) is UGES if (5) is relaxed to

$$\exists \eta^\star > 0 \text{ s.t. } \sup_{\eta \ge \eta^\star} \eta N(\eta) < \frac{K - \lambda}{11D(\Theta - 1)\bar{\bar{c}}}. \tag{28}$$

A similar relaxation can be made in the more general UGAS setting covered by Theorem 4.

5 Proof of Theorem 6

We first set $\dot{U}^{[\alpha]} = U_t^{[\alpha]}(x, t) + U_x^{[\alpha]}(x, t) f(x, t, \alpha t)$ for all $x \in \mathbb{R}^n$, $t \ge 0$, and $\alpha > 0$. Using Assumptions H1. and H3. and (17) with $p(t, l)$ independent of t gives

$$\begin{aligned}\dot{U}^{[\alpha]} \le\ & -W(x,t) + p(\alpha t)\Theta(x,t) - p(\alpha t)\Theta(x,t) \\ & + \left(\int_{t-1}^t p(\alpha l)dl\right)\Theta(x,t) \\ & - \left(\int_{t-1}^t \left(\int_s^t p(\alpha l)dl\right)ds\right)\left(\tfrac{\partial\Theta}{\partial x}f + \tfrac{\partial\Theta}{\partial t}\right) \\ \le\ & -W(x,t) + \left|\int_{t-1}^t p(\alpha l)dl\right| \tfrac{1}{c}W(x,t) \\ & + \left|\int_{t-1}^t \left(\int_s^t p(\alpha l)dl\right)ds\right| \tfrac{1}{c}W(x,t)\end{aligned} \tag{29}$$

along trajectories of (1), where we omit the argument $(x, t, \alpha t)$ of f. For any $\alpha > 0$, $t \ge 0$, and $s \in [\alpha t - \alpha, \alpha t]$, Assumption H2. gives

$$\int_s^{\alpha t} p(l)\mathrm{d}l = \int_s^{\bar{\tau}(s)} p(l)\mathrm{d}l + \int_{\underline{\tau}(\alpha t)}^{\alpha t} p(l)\mathrm{d}l, \tag{30}$$

where $\bar{\tau}(u) := \min\{kT : k \in \mathbb{Z}, kT \ge u\}$ and $\underline{\tau}(u) := \max\{kT : k \in \mathbb{Z}, kT \le u\}$. The proof of (30) uses the fact that the integral of p over the interval $[\min\{\bar{\tau}(s), \underline{\tau}(\alpha t)\}, \max\{\bar{\tau}(s), \underline{\tau}(\alpha t)\}]$ is zero, by H2 and the fact that both endpoints of this interval are integer multiples of T. Choosing $p_{\max}$ to be any global bound on $|p(l)|$ over all $l \in \mathbb{R}$, (30) gives

$$\left|\int_s^{\alpha t} p(l)\mathrm{d}l\right| \le 2Tp_{\max} \quad \forall s \in [\alpha t - \alpha, \alpha t] \tag{31}$$

for all $t \ge 0$ and $\alpha > 0$. Hence, for all $t \ge 0$,

$$\begin{aligned}\left|\int_{t-1}^t p(\alpha l)dl\right| &= \frac{1}{\alpha}\left|\int_{\alpha t - \alpha}^{\alpha t} p(l)dl\right| \le \frac{2Tp_{\max}}{\alpha} \\ \left|\int_{t-1}^t \left(\int_s^t p(\alpha l)dl\right)ds\right| &= \frac{1}{\alpha^2}\left|\int_{\alpha t-\alpha}^{\alpha t}\left(\int_s^{\alpha t} p(l)dl\right)ds\right| \\ &\le \frac{1}{\alpha}\sup_{s\in[\alpha t-\alpha, \alpha t]}\left|\int_s^{\alpha t} p(l)dl\right| \\ &\le 2Tp_{\max}/\alpha\end{aligned} \tag{32}$$

Using these estimates, (29) gives $\dot{U}^{[\alpha]} \le -W(x, t)/2$ for all $x \in \mathbb{R}^n$ and $t \ge 0$, as long as $\alpha > 8Tp_{\max}/c$. Since $W \in$ UPD, this gives the Lyapunov decay estimate. By Assumption H3. and (32), $U^{[\alpha]} \in$ UPPD for large enough constants $\alpha > 0$. This proves the theorem.

6 Illustrations of Theorem 4

We next illustrate how Theorem 4 extends the results of Khalil (2002) and Peuteman & Aeyels (2002). In Peuteman & Aeyels (2002), the limiting dynamics (2) are assumed to be UGES. However, in our first example, the limiting dynamics are UGAS but not necessarily UGES. We next consider a class of systems (1) from Peuteman & Aeyels (2002) that arises in identification where the limiting dynamics (2) is linear and exponentially stable. For these systems, our work extends Peuteman & Aeyels (2002) by providing formulas for Lyapunov functions for (1) that are expressed in terms of the quadratic Lyapunov functions for the limiting dynamics and that have the additional desirable property that they are also ISS Lyapunov functions for (3) for suitable functions g. Finally, we apply our results to a friction model for a mass-spring dynamics. In all three examples, the limiting dynamics has a simple Lyapunov function structure so our results give explicit Lyapunov functions for the original rapidly time-varying dynamics.

6.1 Application to a UGAS dynamics that is not UGES

Consider the following variant of the scalar example on (Peuteman & Aeyels, 2002, p.53):

$$\begin{aligned}&\dot{x} = f(x,t,\alpha t) =\\ &-\sigma_1(x)[2+\sin(t+\cos(\sigma_2(x)))]\{1+10\sin(\alpha t)\}\end{aligned} \tag{33}$$

where $\sigma_1, \sigma_2 : \mathbb{R} \to \mathbb{R}$ are C^1 functions such that σ_1 is odd, $\sup\{|\sigma_1'(x)| + |\sigma_1(x)\sigma_2'(x)| : x \in \mathbb{R}\} < \infty$, $\sigma_1 \in \mathcal{K}$ on $[0,\infty)$, and $\sigma_1''(s) \le 0$ for all $s > 0$. One easily verifies the hypotheses of Theorem 4 using

$$\begin{aligned}&\bar{f}(x,t) := -\sigma_1(x)[2+\sin(t+\cos(\sigma_2(x)))],\\ &V(x,t) \equiv \bar{V}(x) := \int_0^x \sigma_1(s)\mathrm{d}s,\\ &\delta(s) := 33\sigma_1(2s), \text{ and } N(\eta) := 60/\eta^2 \text{ for large } \eta.\end{aligned}$$

This allows e.g. $\sigma_1(s) = \sigma_2(s) = \arctan(s)$ in which case (2) is UGAS but not UGES because $|\dot{x}(t)| \le 2\pi$ along all of its trajectories $x(t)$. Condition P_1 follows because $\sigma_1(2s) \le 2\sigma_1(s)$ for all $s \ge 0$, which holds because $\sigma_1''(s) \le 0$ for all $s \ge 0$. Theorem 5 then gives the following iISS Lyapunov function for (12) for large $\alpha > 0$:

$$\bar{V}\left(\xi + 5\sqrt{\alpha}\,\sigma_1(\xi)\int_{t-\frac{2}{\sqrt{\alpha}}}^t \int_s^t \mu(\xi,l)\sin(\alpha l)\,\mathrm{d}l\,\mathrm{d}s\right), \tag{34}$$

where $\mu(\xi,l) := 2+\sin(l+\cos(\sigma_2(\xi)))$. In particular, this is a Lyapunov function for $\dot{x} = f(x,t,\alpha t)$, and it is also an ISS Lyapunov function for (12) if $\delta \in \mathcal{K}_\infty$ (e.g. if $\sigma_1(s) = \mathrm{sgn}(s)\ln(1+|s|)$ for $|s| \ge 1$ and $\sigma_2(s) = \arctan(s)$). Our conditions on the σ_i's cannot be omitted even if the limiting dynamics is UGES; see (Peuteman & Aeyels, 2002, §8.2). For example, if $\sigma_1(x) = x$ and $\sigma_2(x) = x^2$, then (2) is UGES, but (33) is only shown to be *locally* exponentially stable for large $\alpha > 0$; see Peuteman & Aeyels (2002). This does not contradict our theorem because in that case (14) would be violated.

6.2 A system arising in identification

Consider the following variant of the example in (Peuteman & Aeyels, 2002, Section 8.1):

$$\dot{x} = f(\alpha t)\,m(t)\,m^T(t)\,x + g(x,t,\alpha t)\,u, \tag{35}$$

with state $x \in \mathbb{R}^n$ and inputs $u \in \mathbb{R}^m$, where we assume

i. $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous and admits a $o(s)$ function M and a constant $f^\star < 0$ for which
  (1) $f^\star = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^T f(s)\mathrm{d}s$
  (2) $|\int_{t_1}^{t_2}[f(s)-f^\star]\mathrm{d}s| \le M(t_2-t_1)$ if $t_2 \ge t_1$

ii. $m : \mathbb{R} \to \mathbb{R}^n$ is continuous and satisfies $|m(t)| = 1$ for all $t \in \mathbb{R}$, and there exist constants $\alpha', \beta', \tilde{c} > 0$ such that for all $t \in \mathbb{R}$, $\alpha > 0$, and $x \in \mathbb{R}^n$, we have:
  (1) $\alpha' I \le \int_t^{t+\tilde{c}} m(\tau)m^T(\tau)\mathrm{d}\tau \le \beta' I$
  (2) $||g(x,t,\alpha t)|| \le \beta'\{1+\sqrt{|x|}\}$

iii. $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times m}$ is continuous and is C^1 in x, and there exists a constant $K > 1$ such that $|\partial g_{ij}(x,t,\alpha t)/\partial x| \le K\ \forall x \in \mathbb{R}^n$, $t \ge 0$, and $\alpha > 0$, and each component g_{ij} of g.

Here I denotes the $n \times n$ identity matrix. Notice that we allow f to take both positive and negative values. The special case of (35) where $g \equiv 0$ was studied in Peuteman & Aeyels (2002) where it is shown that the corresponding rapidly varying dynamics $\dot{x} = f(\alpha t)m(t)m^T(t)x$ satisfies the hypotheses of Theorem 4 with the UGES dynamics

$$\dot{x} = \bar{f}(x,t) := f^\star m(t)\,m^T(t)\,x \tag{36}$$

and with δ of the form $\delta(s) = \bar{r}s$ for a constant $\bar{r} > 0$. The particular case of (35) in which $\dot{x} = -m(t)\,m^T(t)\,x$ has been extensively studied in systems identification; see Peuteman & Aeyels (2002). However, these earlier results do not provide explicit ISS Lyapunov functions for (35). On the other hand, the following lemma provides an explicit Lyapunov function for (36):

Lemma 6 *Let assumptions i.-iii. hold. If we choose*

$$P(t) = \kappa I + \int_{t-\tilde{c}}^t \int_s^t m(l)m^T(l)\,\mathrm{d}l\,\mathrm{d}s, \tag{37}$$

where $\kappa = \tilde{c}/(2|f^\star|) + \frac{1}{4\alpha'}\tilde{c}^4|f^\star|$ *then* $V(x,t) = x^T P(t)x$ *is a Lyapunov function for (36) for which* $2V/\alpha'$ *satisfies the requirements of Lemma 1.*

To prove Lemma 6, we apply (17)-(18) with $\tau = \tilde{c}$ and $p(t,l) \equiv m(l)m^T(l)$ and group terms to check that the derivative of V along trajectories of (36) satisfies

$$\begin{aligned}\dot{V} &= (2f^\star\kappa+\tilde{c})x^T m(t)m^T(t)x\\ &\quad +2f^\star x^T\left[\int_{t-\tilde{c}}^t\int_s^t m(l)m^T(l)\,\mathrm{d}l\,\mathrm{d}s\right]m(t)m^T(t)x\\ &\quad -x^T\left[\int_{t-\tilde{c}}^t m(l)m^T(l)\,\mathrm{d}l\right]x\\ \dot{V} &\le (2f^\star\kappa+\tilde{c})|m^T(t)x|^2-\alpha'|x|^2\\ &\quad +2|f^\star||x|\left[\int_{t-\tilde{c}}^t\int_s^t|m(l)|^2\mathrm{d}l\mathrm{d}s\right]|m(t)||m^T(t)x|\\ &\le (2f^\star\kappa+\tilde{c})|m^T(t)x|^2+\tilde{c}^2|f^\star|\,|x||m^T(t)x|-\alpha'|x|^2\end{aligned}$$

everywhere, by ii.(1). By the triangle inequality,

$$\tilde{c}^2|f^\star||x||m^T(t)x| \le \frac{1}{2}\alpha'|x|^2 + \frac{1}{2\alpha'}\tilde{c}^4|f^\star|^2|m^T(t)x|^2$$

so $\dot{V} \le \left(2f^\star\kappa+\tilde{c}+\frac{1}{2\alpha'}\tilde{c}^4|f^\star|^2\right)|m^T(t)x|^2 - \frac{1}{2}\alpha'|x|^2$ holds everywhere. Recalling that $|m(t)| = 1$ everywhere and that P is everywhere positive definite, the fact that $2V/\alpha'$ satisfies the requirements of Lemma 1 follows immediately from our choice of κ. Remark 4 now gives:

Corollary 7 *Let (35) satisfy conditions i.-iii. above and let V be as in Lemma 6. Then there exists a constant $\alpha_o > 0$ such that for each constant $\alpha > \alpha_o$,*

$$V^{[\alpha]}(x,t) := V\left(\left[I - \frac{\sqrt{\alpha}}{2}\int_{t-2/\sqrt{\alpha}}^{t}\int_{s}^{t} D(l)\,\mathrm{d}l\,\mathrm{d}s\right]x, t\right)$$

where $D(l) = (f(\alpha l) - f^\star)m(l)m^T(l)$ is an ISS Lyapunov function for (35).

6.3 Friction example

The following one degree-of-freedom mass-spring system from de Queiroz *et al.* (2000) arises in the control of mechanical systems in the presence of friction:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\sigma_1(\alpha t)x_2 - k(t)x_1 + u\\ &\quad - \left\{\sigma_2(\alpha t) + \sigma_3(\alpha t)e^{-\beta_1\mu(x_2)}\right\}\mathrm{sat}(x_2)\end{aligned} \tag{38}$$

where x_1 and x_2 are the mass position and velocity, respectively; σ_i, $i = 1,2,3$ denote positive time-varying viscous, Coulomb, and static friction-related coefficients, respectively; β_1 is a positive constant corresponding to the Stribeck effect; $\mu(\cdot)$ is a positive definite function also related to the Stribeck effect; k denotes a positive time-varying spring stiffness-related coefficient; and $\mathrm{sat}(\cdot)$ denotes any continuous function having these properties:

$$\begin{array}{ll}\text{(a)}\ \mathrm{sat}(0) = 0, & \text{(b)}\ \xi\,\mathrm{sat}(\xi) \geq 0\ \ \forall \xi \in \mathbb{R},\\ \text{(c)}\ \lim_{\xi\to+\infty}\mathrm{sat}(\xi) = +1, & \text{(d)}\ \lim_{\xi\to-\infty}\mathrm{sat}(\xi) = -1\end{array} \tag{39}$$

We model the saturation as the differentiable function

$$\mathrm{sat}(x_2) = \tanh(\beta_2 x_2), \tag{40}$$

where β_2 is a large positive constant. Note for later use that $|\mathrm{sat}(x_2)| \leq \beta_2|x_2|$ for all $x_2 \in \mathbb{R}$. We assume the friction coefficients vary in time faster than the spring stiffness coefficient so we restrict to cases where $\alpha > 1$.

Our precise mathematical assumptions on (38) are: k and the σ_i's are bounded C^1 functions; μ has a globally bounded derivative; and there exist constants $\tilde{\sigma}_i$, with $\tilde{\sigma}_1 > 0$ and $\tilde{\sigma}_i \geq 0$ for $i = 2,3$, and a $o(s)$ function $s \mapsto M(s)$ such that

$$\left|\int_{t_1}^{t_2}(\sigma_i(t) - \tilde{\sigma}_i)\,\mathrm{d}t\right| \leq M(t_2 - t_1), \quad i = 1,2,3 \tag{41}$$

for all $t_1, t_2 \in \mathbb{R}$ satisfying $t_2 > t_1$. Although the σ_i's are positive for physical reasons, we will not require their positivity in the sequel. We show (38) satisfies the requirements of the version of Theorem 4 from Remark 4 (with $\delta(s) = \bar{r}s$ for a constant $\bar{r}$) when (2) is

$$\begin{aligned}&\dot{x}_1 = x_2\\ &\dot{x}_2 = -\tilde{\sigma}_1 x_2 - \left\{\tilde{\sigma}_2 + \tilde{\sigma}_3 e^{-\beta_1\mu(x_2)}\right\}\mathrm{sat}(x_2) - k(t)x_1\,,\end{aligned} \tag{42}$$

assuming this additional condition whose physical interpretation is that the spring stiffness is nonincreasing:

$$\exists k_o, \bar{k} > 0 \ \text{ s.t. }\ k_o \leq k(t) \leq \bar{k} \ \text{ and }\ k'(t) \leq 0 \ \ \forall t \geq 0.$$

To this end, set $S := \tilde{\sigma}_1 + (\tilde{\sigma}_2 + \tilde{\sigma}_3)\beta_2$ and

$$V(x,t) = A(k(t)x_1^2 + x_2^2) + x_1x_2, \tag{43}$$

where $A := 1 + 1/k_o + [1 + S^2/k_o]/\tilde{\sigma}_1$. Since $A\bar{k} \geq 1$, $\frac{1}{2}(x_1^2 + x_2^2) \leq V(x,t) \leq A^2\bar{k}(|x_1| + |x_2|)^2$ for all $x \in \mathbb{R}^2$ and $t \geq 0$. Also, since $k' \leq 0$ everywhere, the derivative $\dot{V} = V_t(x,t) + V_x(x,t)\bar{f}(x,t)$ along trajectories of (42) satisfies

$$\begin{aligned}\dot{V} \leq V_x(x,t)\bar{f}(x,t) &= [2Ak(t)x_1 + x_2]x_2 - [2Ax_2 + x_1]\\ &\quad\times\{\tilde{\sigma}_1 x_2 + \left[\tilde{\sigma}_2 + \tilde{\sigma}_3 e^{-\beta_1\mu(x_2)}\right]\mathrm{sat}(x_2) + k(t)x_1\}\end{aligned}$$

and therefore, by grouping terms, we also have

$$\begin{aligned}&\dot{V} \leq -k_0x_1^2 - (2A\tilde{\sigma}_1 - 1)x_2^2 + S|x_1x_2| \quad \text{(by (39)(b))}\\ &\leq -b|x|^2 - \left[\tfrac{k_o}{2}x_1^2 + (A\tilde{\sigma}_1 - 1/2)x_2^2 - S|x_1x_2|\right]\\ &= -b|x|^2 - \tfrac{k_o}{2}\left(|x_1| - \tfrac{S}{k_o}|x_2|\right)^2 + \left(\tfrac{S^2}{2k_o} + \tfrac{1}{2} - A\tilde{\sigma}_1\right)x_2^2\\ &\leq -b|x|^2, \text{ where } b := \min\{k_o/2, A\tilde{\sigma}_1 - 1/2\}.\end{aligned}$$

The preceding inequalities imply that V/b is a Lyapunov function for (42) satisfying the requirements of Lemma 1. The integral bound requirement on (13) from our theorem follows from (41) and the sublinear growth of tanh, since the integral bound can be verified term by term. The remaining bounds from (14) follow because μ and sat have globally bounded derivatives. We conclude that for sufficiently large constants $\alpha > 0$, (38) admits the ISS Lyapunov function

$$V^{[\alpha]}(\xi,t) = V\left(\xi_1, \xi_2 + \tfrac{\sqrt{\alpha}}{2}\int_{t-\frac{2}{\sqrt{\alpha}}}^{t}\int_s^t \Gamma_\alpha(l,\xi)\,\mathrm{d}l\,\mathrm{d}s, t\right)$$

where V is the Lyapunov function (43) for (42) and

$$\Gamma_\alpha(l,\xi) := \{\sigma_1(\alpha l) - \tilde{\sigma}_1\}\xi_2 + \mu_\alpha(l,\xi)\tanh(\beta_2\xi_2) \tag{44}$$

$$\mu_\alpha(l,\xi) := \sigma_2(\alpha l) - \tilde{\sigma}_2 + (\sigma_3(\alpha l) - \tilde{\sigma}_3)e^{-\beta_1\mu(\xi_2)} \tag{45}$$

so (38) is ISS for large enough $\alpha > 0$, by Remark 4.

Remark 8 The preceding construction simplifies considerably if σ_2 and σ_3 in (38) are positive constants. In that case, the limiting dynamics (2) can be taken to be

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tilde{\sigma}_1 x_2 - \left\{\sigma_2 + \sigma_3 e^{-\beta_1\mu(x_2)}\right\}\text{sat}(x_2) - k(t)x_1\end{aligned}$$

and the ISS Lyapunov function for (38) becomes

$$V\left(\xi_1, \xi_2\left(1 + \frac{\sqrt{\alpha}}{2}\int_{t-\frac{2}{\sqrt{\alpha}}}^{t}\int_s^t\{\sigma_1(\alpha l) - \tilde{\sigma}_1\}\,\mathrm{d}l\,\mathrm{d}s\right), t\right)$$

with V defined by (43), since the $\mu_\alpha(l,\xi)\tanh(\beta_2\xi_2)$ term in the difference $f - \bar{f}$ is no longer present.

7 Illustrations of Theorem 6

Simple calculations show that Theorem 6 applies to (33) and (35) without controls (with the same choices of V that we used in our earlier discussions of those systems), assuming in the latter case that requirements i.-iii. hold and for instance f is a suitable periodic function and $\dot{m}(t)$ is bounded. We next show how Theorem 6 also applies to cases that are not tractable by Theorem 4.

7.1 Dynamics that are not globally Lipschitz

Simple calculations that we omit because of space constraints show that the one-dimensional dynamics

$$\dot{x} = -x^3 + 10\cos(\alpha t)\frac{x^3}{1+x^2} \tag{46}$$

satisfy H. using $V(x,t) \equiv x^4/4$, $W(x,t) \equiv x^6$, $\Theta(x,t) \equiv x^6/(1+x^2)$, $p(t) = 10\cos(t)$, $T = 2\pi$, and small enough $c > 0$, so (46) has the global Lyapunov function

$$U^{[\alpha]}(x,t) = \frac{x^4}{4} - \left(\int_{t-1}^{t}\left(\int_s^t 10\cos(\alpha l)dl\right)ds\right)\frac{x^6}{1+x^2}$$

when $\alpha > 0$ is large enough. However, (46) is not covered by Theorem 4 since it is not globally Lipschitz in x.

7.2 Dynamics with unknown functional parameters

Consider the nonautonomous scalar system

$$\dot{x} = p(\alpha t)\frac{x^2}{1+x^2} + u, \tag{47}$$

where p is an unknown, fast time-varying parameter satisfying $|p(l)| \leq a_m$ for all l and some constant $a_m > 0$ and admitting a constant $T > 0$ such that Assumption H2. holds for all $k \in \mathbb{Z}$. We assume the control u is amplitude limited in the sense that $|u| \leq u_m$ for some constant $u_m > 0$. We show that the saturated state feedback

$$u = -u_m \arctan(x) \tag{48}$$

renders (47) UGAS. (A similar argument shows that $u = -u_m \arctan(Rx)$ stabilizes (47) for any constant $R > 1$.) For simplicity, let $a_m = 10$ and $u_m = 2$. The derivative of $V(x) = x^2/2$ along (47) in closed loop with (48) is

$$\dot{V} = -2x\arctan(x) + p(\alpha t)\frac{x^3}{1+x^2}. \tag{49}$$

Simple calculations allow us to verify the hypotheses of Theorem 6 for the closed loop system using $W(x,t) \equiv 2x\arctan(x)$, $\Theta(x,t) \equiv x^3/(1+x^2)$, and small enough $c > 0$, so we know (48) indeed uniformly globally asymptotically stabilizes (47) with control Lyapunov function

$$U^{[\alpha]}(x,t) = \frac{1}{2}x^2 - \left(\int_{t-1}^{t}\int_s^t p(\alpha l)\,\mathrm{d}l\,\mathrm{d}s\right)\frac{x^3}{1+x^2} \tag{50}$$

when $\alpha > 0$ is sufficiently large.

8 Conclusions

The main hypotheses of (Peuteman & Aeyels, 2002, Theorem 3) are sufficient for uniform global (rather than just local) exponential stability of rapidly time-varying nonlinear systems. We provided complementary results to Peuteman & Aeyels (2002) by establishing uniform global asymptotic (but not necessarily exponential) stability of fast time-varying dynamics without requiring exponential stability of the limiting dynamics, and by constructing global Lyapunov functions for fast time-varying systems. Our Lyapunov constructions are new even in the special case where the dynamics are exponentially stable, and are input-to-state stable Lyapunov functions when the dynamics are control affine, under appropriate conditions. Our results apply to dynamics that are not necessarily uniformly Lipschitz in the state. We illustrated our methods using a friction example.

References

Angeli, D., Sontag, E.D., & Wang, Y. (2000). A characterization of integral input to state stability. *IEEE Trans. Automatic Control*, 45(6), 1082-1097.

Bacciotti, A., & Rosier, L. (2005). *Liapunov Functions and Stability in Control Theory, 2nd Edition.* New York: Springer Verlag.

de Queiroz, M.S., Dawson, D.M., Nagarkatti, S., & Zhang, F. (2000). *Lyapunov-Based Control of Mechanical Systems.* Cambridge, MA: Birkhäuser.

Edwards, H., Lin, Y., & Wang, Y. (2000). On input-to-state stability for time-varying nonlinear systems. *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, (pp. 3501-3506).

Grüne, L., Sontag, E.D., & Wirth, F. (1999). Asymptotic stability equals exponential stability, and ISS equals finite energy gain - if you twist your eyes. *Systems Control Lett.*, 38(2), 127-134.

Hale, J. K. (1980). *Ordinary Differential Equations.* Malabor, FL: Krieger.

Khalil, H. (2002) *Nonlinear Systems, Third Edition.* Englewood Cliffs, NJ: Prentice Hall.

Malisoff, M., & Mazenc, F. (2005). Further remarks on strict input-to-state stable Lyapunov functions for time-varying systems. *Automatica*, 41(11), 1973-1978.

Mazenc, F., & Bowong, S. (2004). Backstepping with bounded feedbacks for time-varying systems. *SIAM Journal on Control and Optimization*, 43(3), 856-871.

Peuteman, J., & Aeyels, D. (2002). Exponential stability of nonlinear time-varying differential equations and partial averaging. *Mathematics of Control, Signals, and Systems*, 15(1), 42-70.

Sontag, E.D. (1989). Smooth stabilization implies coprime factorization. *IEEE Trans. Automatic Control*, 34(4), 435-443.

Sontag, E.D. (1998). Comments on integral variants of ISS. *Systems Control Lett.*, 34(1-2), 93-100.

Sontag, E.D., & Wang, Y. (1995). On characterizations of the input-to-state stability property. *Systems Control Lett.*, 24(5), 351-359.