Global Asymptotic Stabilization for Chains of Integrators with a Delay in the Input

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Abstract: The problem of the global uniform asymptotic stabilization by bounded feedback of a chain of integrators with a delay in the input is solved. No limitation on the size of the delay is imposed. To validate the approach, a third order example is presented.

Keywords. time-delay systems, bounded feedback, chain of integrators.

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1 Introduction

Linear systems with delays in the input have been extensively studied in the last decades [4], [1]. In particular, the stabilization of chains of integrators with delayed control has been a challenging case study for researchers in this field, because of robustness problems with respect to the delay values [6]. Control strategies that involve distributed delay control laws based on the ideas of the Smith predictor, [8], [2] give good theoretical solutions but present in some cases realization problems [15], [9]. Moreover, the control laws obtained that way are not bounded in norm. On the other hand, simple proportional or proportional-integral controllers are sensitive with respect to the delay bounds if an integrator belongs to the loop (see, for instance, [6] and the references therein).

In conclusion, it follows that some natural questions arise: is it possible to construct feedbacks bounded in norm which globally uniformly asymptotically stabilize chains of integrators with delay in the input ? How robust is the closed-loop in terms of delays ? The study of these problems is motivated by the fact that, first, it is appealing from a practical point of view to have at our disposal bounded feedbacks and, second, computational or propagation delays in the input are frequently encountered in practice. Furthermore, in some cases (oscillating systems), delays can be induced in a system in order to guarantee closed-loop stability.

Before 1992, the determination of explicit expressions of globally asymptotically stabilizing feedbacks for chains of integrators of dimension strictly larger than two without delay was an open problem. A. Teel in [11, 12] has solved it by exhibiting a simple family of stabilizing feedbacks for these systems. This pioneer result has been very fruitful. First, it allowed to solve theoretical issues (see [13, 10]) and more applied problems (see [14]) and, second, it is at the origin of a new control design methodology called forwarding (see for example [5, 3]).

Since the forwarding approach yields bounded feedbacks for feedforward systems and in particular for chains of integrators when no delay is present, it is natural to investigate whether this technique can be extended to the case where there is a delay in the input.

This is the purpose of the present paper. More precisely, we solve the problem of stabilizing a chain of integrators of arbitrary length with a delay in the input of arbitrary size using feedbacks arbitrarily small in norm.

Although we conjecture that both the approaches of [5] and [3] can be adapted to the problem we consider, we have chosen to exploit the results in [11, 12]. It turns out that the technique of proof of these works leads to a reasonably simple analysis.

The strategy of proof consists in showing that we can select in a family of feedbacks very close to the one proposed in [11] some elements which globally uniformly asymptotically stabilize the n-dimensional chain of integrators.

The system under consideration is introduced in Section 1 and some key results are presented in Section 2. The main result is stated and proved in Section 3. A third order example is proposed in Section 4. Some concluding remarks in Section 5 end the paper.

1.1 The system and the control law

We consider the chain of integrators of dimension n with delay τ in the input. This system is described by the equations

$$\dot{y}_i(t) = y_{i+1}(t), i = 1, \ldots, n-1 \ , \tag{1}$$

$$\dot{y}_n(t) = u(t-\tau) \ , \tag{2}$$

where $y_i \in R$, $u \in R$ is the input and $\tau \geq 0$ is the delay.

The bounded feedback control considered in this paper has the general form

$$u(t-\tau) = u_S(y_1(t-\tau), y_2(t-\tau), \ldots, y_n(t-\tau))$$

and extensively uses, in a manner that will be explained later, the following class of saturations:

Definition 1 *Let M be a strictly positive constant. We call M-saturation the function $\sigma : R \longrightarrow R$ defined by*

a- $\sigma(s) = s$ when $|s| < M$

b- $\sigma(s) = M$ when $s \geq M$, $\sigma(s) = -M$ when $s \leq -M$

Throughout the paper, the time argument of the function is dropped, whenever no confusion can arise from the context. We denote respectively $\xi(t), \xi(t-\tau)$ and $[\xi(t-\tau) - \xi(t)]$ by $\xi, \bar{\xi}, \tilde{\xi}$.

2 Preliminary results

In this section, we give a time rescaling homogeneous transformation and a linear transformation of the chain of integrators which simplify the analysis of the problem we address in Section 3.

2.1 Homogeneous property

A key homogeneous property of the system (1), (2) is summarized in the following technical lemma:

Lemma 1 *Assume that there exists a feedback bounded in norm by a constant $B > 0$,*

$$u_{Sw}(w_1, w_2, ..., w_n)$$

which globally uniformly asymptotically stabilizes the system

$$\dot{w}_i(t) = w_{i+1}(t), i = 1, \ldots, n-1 \ , \tag{3}$$

$$\dot{w}_n(t) = u(t-\theta) \ , \tag{4}$$

where θ is a constant delay. Then, for all real number $k > 0$, the control law bounded in norm by $k^n B$

$$u_{Sv}(v_1, v_2, ..., v_n) = k^n u_{Sw}(v_1, \ldots, k^{-n+1} v_n)$$

globally uniformly asymptotically stabilizes the n-dimensional chain of integrators with $\frac{\theta}{k}$ as delay in the input:

$$\dot{v}_i(t) = v_{i+1}(t), i = 1, \dots, n-1 \ ,$$
$$\dot{v}_n(t) = u(t - \frac{\theta}{k}) \ .$$

Remark 1 *Roughly speaking, Lemma 1 points out the fact that if one can stabilize a chain of integrators by bounded feedback when the delay in the input is small, then one can also stabilize by bounded feedback the same chain of integrators when there is an arbitrarily large delay in the input.*

Proof. Let $u_{Sw}(w_1, w_2, ..., w_n)$ be a feedback bounded in norm by $B > 0$, which globally uniformly asymptotically stabilizes the system (3), (4). The time rescaling $t = ks$ and the new variables

$$v_i(s) = k^{i-1} w_i(ks) \ , \ i = 1, \dots, n \ ,$$

transform the system (1), (2) into

$$\dot{v}_i(s) = k^i \dot{w}_i(ks) = k^i w_{i+1}(ks) = v_{i+1}(s), i = 1, \dots, n-1 \ ,$$

$$\begin{aligned} \dot{v}_n(s) &= k^n \dot{w}_n(ks) \\ &= k^n u_{Sw}(w_1(ks - \theta), \dots, w_n(ks - \theta)) \\ &= k^n u_{Sw}(v_1(s - \tfrac{\theta}{k}), \dots, k^{-n+1} v_n(s - \tfrac{\theta}{k})) \ . \end{aligned}$$

Denoting

$$u_{Sv}(v_1,\dots,v_n) := k^n u_{S_w}(v_1,\dots,k^{-n+1}v_n)$$

the previous system becomes

$$\begin{aligned}
\dot{v}_i(s) &= v_{i+1}(s), i = 1,\dots,n-1 \ ,\\
\dot{v}_n(s) &= u_{Sv}(v_1(s-\frac{\theta}{k}),\dots,v_n(s-\frac{\theta}{k})) \ ,
\end{aligned}$$

which is a globally uniformly asymptotically stable system. ■

Remark 2 *Time rescaling and its advantages in analyzing nonlinear delay systems are known in the literature. To the authors best knowledge this technique was not used so far for controlling delay systems.*

2.2 A specific linear transformation

In order to facilitate the analysis of the stabilizability properties by bounded feedback of the chain of integrators, a linear transformation, $x = Ty$, is carried out in [11]. We recall briefly this result that we will use in the next section.

Proposition 2 *[11] There exists a linear change of coordinates characterized by*

$$x_{n-i} = \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} y_{n-j} \ = h_{n-i}(y) \ , \tag{5}$$

whose inverse transformation is

$$y_{n-i} \ = \sum_{j=0}^{i} (-1)^{i+j} \frac{i!}{j!(i-j)!} x_{n-j} \ ,$$

which transforms the system (1), (2) into

$$\dot{x}_i(t) = \sum_{j=i+1}^{n} x_j(t) + u(t-\tau), i = 1, \dots , n-1 \ , \tag{6}$$

$$\dot{x}_n \ (t) = u(t-\tau) \ . \tag{7}$$

3 Main result

3.1 Technical lemma

We now give a preliminary lemma that is crucial in establishing the main results.

Lemma 3 *Consider the system*

$$\dot{x} = \tilde{z} - \sigma_b(\bar{x} + \sigma_a(\lambda_1(t))) \ , \tag{8}$$

$$\dot{z} = f(z, \tilde{z}, \lambda_2(t)) \ , \tag{9}$$

where $\sigma_a(\cdot)$ and $\sigma_b(\cdot)$ are respectively M_a and M_b-saturations, $f(\cdot)$ is a continuous function and $\lambda_1(t), \lambda_2(t)$ are any continuous functions of t. Assume that, for all continuous function $\lambda_2(t)$, the solutions of the z-subsystem are defined for all $t \geq 0$ and that

A1. The inequality $\frac{1}{8}M_b \geq M_a$ is satisfied.

A2. For all constant $H \geq 0$, there exists a positive constant $\hat{f}_H$ such that, for all continuous function $\lambda_2(t)$, the inequality $\sup_{\{|\alpha| \leq H|, |\beta| \leq H, 0 \leq t\}} |f(\alpha, \beta, \lambda_2(t))| \leq \hat{f}_H$ is satisfied.

A3. There exist $T_c \geq 0$ and a constant $B_c > 0$ so that for all continuous function $\lambda_2(t)$ and for all $t \geq T_c$,the inequality $|z(t)| \leq B_c$ is satisfied.

Then for any continuous initial condition defined on $[-\tau, 0]$, the solution is defined for all $t \geq 0$ and if $\tau \in \left[0, \min\left\{\frac{M_b}{8\hat{f}_{B_c}}, \frac{1}{9}\right\}\right]$ then there exists $T_b \geq T_c$ such that

$$|x| \leq \frac{1}{2}M_b \ , \ |\tilde{x}| \leq \frac{1}{8}M_b \ , \ \forall t \geq T_b \ . \tag{10}$$

Remark 3 *The property (10) and A1 ensure that any solution eventually enters the domain where the saturation function $\sigma_b(\cdot)$ operates in its linear region.*

Proof. The fact that all the solutions are defined for all $t \geq 0$ is obvious. Indeed, any solution can be continued "step-by-step" from one delay interval to the next, as the solution of a standard ordinary differential equation. Next, let us prove that when t is sufficiently large, $|\tilde{x}|$ is smaller than a function of τ which converges to zero when τ converges to zero.

According to A3, for all $t \geq T_c + \tau$, $|z(t)| \leq B_c$ and $|z(t-\tau)| \leq B_c$. So, according to A2 we have

$$|\dot{z}| \ \leq \ \hat{f}_{B_c} \ , \ \forall t \geq T_c + \tau \ .$$

It straightforwardly follows that

$$|\tilde{z}| \ := \ |z(t) - z(t-\tau)| \leq \tau \hat{f}_{B_c} \ , \ \forall t \geq T_c + \tau \ . \tag{11}$$

On the other hand,

$$|\tilde{x}| \ := \ |x(t-\tau) - x(t)| \leq \tau \sup_{s \in [t-\tau, t]} (|\tilde{z}(s)| + M_b) \ . \tag{12}$$

Combining (11), (12), we deduce

$$|\tilde{x}| \;\leq\; \tau[\tau \hat{f}_{B_c} + M_b] \;,\; \forall t \geq T_c + 2\tau \;.$$

The next step consists in analyzing the evolution of the variable $x(t)$ with the help of the Lyapunov function $V(x) = \frac{1}{2}x^2$. Its derivative along the trajectories of (8) satisfies

$$\begin{aligned}\dot{V} &= x[\tilde{z} - \sigma_b(x + \tilde{x} + \sigma_a(\lambda_1(t)))] \\ &\leq \tau \hat{f}_{B_c}|x| - x\sigma_b(x + \tilde{x} + \sigma_a(\lambda_1(t))) \;,\; \forall t \geq T_c + 2\tau \;.\end{aligned}$$

We know from (12) and condition b of Definition 1 that, for all $t \geq T_c + 2\tau$,

$$|x + \tilde{x} + \sigma_a(\lambda_1(t))| \;\geq\; |x| - |\tilde{x}| - |\sigma_a(\lambda_1(t))| \;\geq\; |x| - \tau(\tau \hat{f}_{B_c} + M_b) - M_a \;. \tag{13}$$

When $\tau \in \left[0, \min\left\{\frac{M_b}{8\hat{f}_{B_c}}, \frac{1}{9}\right\}\right]$ then

$$\tau(\tau \hat{f}_{B_c} + M_b) \;\leq\; \frac{1}{8}M_b \;. \tag{14}$$

When the condition (14) is satisfied, then, for all $t \geq T_c + 2\tau$,

$$|x + \tilde{x} + \sigma_a(\lambda_1(t))| \;\geq\; |x| - \frac{1}{4}M_b \;. \tag{15}$$

It follows that when $|x(t)| \geq \frac{1}{2}M_b$ with $t \geq T_c + 2\tau$,

$$\dot{V} \;\leq\; \tau \hat{f}_{B_c}|x| - |x|\sigma_b(\frac{1}{4}M_b) \;=\; (\tau \hat{f}_{B_c} - \frac{1}{4}M_b)|x| \;. \tag{16}$$

When $\tau \in \left[0, \min\left\{\frac{M_b}{8\hat{f}_{B_c}}, \frac{1}{9}\right\}\right]$ then $\tau \hat{f}_{B_c} \leq \frac{1}{8}M_b$. So when $|x(t)| \geq \frac{1}{2}M_b$ with $t \geq T_c + 2\tau$, the inequalities

$$\dot{V} \;\leq\; -\frac{1}{8}M_b|x| \;\leq\; -\frac{1}{16}M_b^2 \tag{17}$$

are satisfied. It follows from this reasoning that there exists $T_b \geq T_c + 2\tau$ such that for all $t \geq T_b$, $|x(t)| \leq \frac{1}{2}M_b$. Moreover, according to (14), for all $t \geq T_b$, $|\tilde{x}| \leq \frac{1}{8}M_b$. This concludes the proof. ■

3.2 Stability for small delays

We are now ready to state a first stabilizability result for the system (6), (7).

Theorem 4 *The bounded control law*

$$u_{Sx}(t-\tau) = -\sigma_n(x_n(t-\tau) + \sigma_{n-1}(x_{n-1}(t-\tau) + \ldots + \sigma_2(x_2(t-\tau) + \sigma_1(x_1(t-\tau)))\ldots)) \ , \tag{18}$$

where each $\sigma_i(\cdot)$ is a M_i-saturation and $M_i \leq \frac{1}{8}M_{i+1}, i = 1, \ldots, n-1$, globally uniformly asymptotically stabilizes the system (6), (7) for delays τ sufficiently small.

Remark 4 *The functions (18) are continuous but not continuously differentiable. We have chosen to utilize the functions $\sigma_i(\cdot)$ because they lead to a simple analysis. However, one can prove a similar result with smooth saturations.*

The proof of Theorem 4 follows from the two results stated below. Lemma 5 says that all the solutions of the closed-loop system are bounded and eventually enter in the domain where all the saturations operate linearly. Lemma 6 establishes the asymptotic stability of the linear approximation of the closed-loop system.

Lemma 5 *Consider the bounded control law*

$$u_{Sx}(t-\tau) \;=\; -\sigma_n(x_n(t-\tau)+\sigma_{n-1}(x_{n-1}(t-\tau)+\ldots+\sigma_2(x_2(t-\tau)+\sigma_1(x_1(t-\tau)))\ldots)) \tag{19}$$

where the M_i-saturations $\sigma_i(\cdot)$ are such that $M_i \le \frac{1}{8}M_{i+1}, i=1,\ldots,n-1$ and τ is such that $0 \le \tau \le \breve{\tau}_n$ with

$$\breve{\tau}_n \;=\; \min\left\{\frac{M_1}{8\left(nM_1+\sum_{j=2}^{n}\frac{M_j}{2}\right)},\frac{1}{9}\right\} \;. \tag{20}$$

Consider any trajectory $x(t)$ of the system (6), (7) in closed-loop with the control (19). Then there exists $T_1 \ge 0$ such that when $t \ge T_1$, then $x(t-\tau)$ is in the domain where the M_i-saturations operate in their linear region: for all $t \ge T_1$, the trajectory is solution of the linear equations

$$\dot{x}_i \;=\; \sum_{j=i+1}^{n} x_j - \sum_{j=1}^{n}\bar{x}_j, i=1,...,n \;. \tag{21}$$

Proof. Observe that the system (6), (7) in closed-loop with the control (18) is described by

$$\dot{x}_i = \sum_{j=i+1}^{n} x_j - \sigma_n(x_n+\tilde{x}_n+\sigma_{n-1}(\bar{x}_{n-1}+\ldots+\sigma_2(\bar{x}_2+\sigma_1(\bar{x}_1))\ldots)).,i=1,\ldots,n-1\;, \tag{22}$$

$$\dot{x}_n = -\sigma_n(x_n+\tilde{x}_n+\sigma_{n-1}(\bar{x}_{n-1}+\ldots+\sigma_2(\bar{x}_2+\sigma_1(\bar{x}_1))\ldots))\;. \tag{23}$$

Observe that the solutions of the closed-loop system are defined for all $t \ge 0$. Next, we prove the result by induction.

Induction hypothesis. There exists $T_{k+1} \geq 0$ such that for all $t \geq T_{k+1}$, the inequalities $|x_j| \leq \frac{M_j}{2}, |\tilde{x}_j| \leq \frac{M_j}{8}, j = k+1, \dots, n$ are satisfied.

Step 1. We analyze the evolution of $x_n(t)$. Using arguments invoked to prove Lemma 3, one can easily demonstrate that, provided that $\tau \leq \frac{1}{9}$, there exists $T_n \geq 0$ so that, for all $t \geq T_n$, then $|x_n| \leq \frac{1}{2}M_n, |\tilde{x}_n| \leq \frac{1}{8}M_n$. So when $t \geq T_n$ the saturation $\sigma_n(\cdot)$ operates in its linear region. More precisely, when $t \geq T_n$ the system is described by

$$\dot{x}_i = \tilde{x}_n + \sum_{j=i+1}^{n-1} x_j - \sigma_{n-1}((\bar{x}_{n-1} + \dots + \sigma_2(\bar{x}_2 + \sigma_1(\bar{x}_1))\dots) \ , \ i = 1, \dots, n-1 \ ,$$

$$\dot{x}_n = -\bar{x}_n - \sigma_{n-1}((\bar{x}_{n-1} + \dots + \sigma_2(\bar{x}_2 + \sigma_1(\bar{x}_1))\dots) \ .$$

The induction hypothesis is satisfied.

Step $n-k$. Assume now that the induction hypothesis is satisfied at the step $n-k$. Then the solution has reached the domain where the saturations $\sigma_j(\cdot), j = k+1, \dots, n$ are linear. Therefore, for all $t \geq T_{k+1}$, we have

$$\dot{x}_l = \sum_{j=l+1}^{k} x_j + \sum_{j=k+1}^{n} \tilde{x}_j - \sigma_k((\bar{x}_k + \dots + \sigma_2(\bar{x}_2 + \sigma_1(\bar{x}_1))\dots) \ , \ l = 1, \dots, k-1 \ ,$$

$$\dot{x}_k = -\sum_{j=k+1}^{n} \tilde{x}_j - \sigma_k((\bar{x}_k + \dots + \sigma_2(\bar{x}_2 + \sigma_1(\bar{x}_1))\dots) \ ,$$

$$\dot{x}_i = -\sum_{j=k+1}^{i} \bar{x}_j - \sum_{j=i+1}^{n} \tilde{x}_j - \sigma_k((\bar{x}_k + \dots + \sigma_2(\bar{x}_2 + \sigma_1(\bar{x}_1))\dots) \ , \ i = k+1, \dots, n-1 \ ,$$

$$\dot{x}_n = -\sum_{j=k+1}^{n} \bar{x}_j - \sigma_k((\bar{x}_k + \dots + \sigma_2(\bar{x}_2 + \sigma_1(\bar{x}_1))\dots) \ .$$

To apply Lemma 3, define

$$z := -\sum_{j=k+1}^{n} x_j \ .$$

Observe that the x_k-subsystem rewrites

$$\dot{x}_k \ = \ \tilde{z} - \sigma_k((\bar{x}_k + \ldots + \sigma_2(\bar{x}_2 + \sigma_1(\bar{x}_1))\ldots) \ .$$

The induction hypothesis ensures that the solution of the z-system is, for all $t \geq T_{k+1}$, in a compact set and that $|\dot{z}(t)| \leq \hat{f}_k$ for all $t \geq T_{k+1}$ with $\hat{f}_k = nM_k + \sum_{j=k+1}^{n} \frac{M_j}{2} > 0$. Therefore, Lemma 3 applies: when

$$\tau \in \left[0, \min_{k\in\{1,\ldots,n-1\}} \left\{ \frac{M_k}{8\left(nM_k + \sum_{j=k+1}^{n} \frac{M_j}{2}\right)}, \frac{1}{9} \right\}\right]$$

there exists $T_k \geq T_{k+1}$ such that, for all $t \geq T_k$, $|x_k| \leq \frac{M_k}{2}, |\tilde{x}_k| \leq \frac{M_k}{8}$, which implies that the domain where $\sigma_k(\cdot)$ is linear is reached. The induction hypothesis is satisfied at the step $n - k + 1$.

Final step. First observe that using the inequalities $M_i \leq \frac{1}{8}M_{i+1}$, $i = 1, \ldots, n-1$ one can easily prove that $\min_{k\in\{1,\ldots,n-1\}} \left\{ \frac{M_k}{8\left(nM_k + \sum_{j=k+1}^{n} \frac{M_j}{2}\right)}, \frac{1}{9} \right\} = \min \left\{ \frac{M_1}{8\left(nM_1 + \sum_{j=2}^{n} \frac{M_j}{2}\right)}, \frac{1}{9} \right\}$. Next, observe that there exists $T_1 \geq 0$ such that when $0 \leq \tau \leq \breve{\tau}_n$ with $\breve{\tau}_n$ given in (20) then for all $t \geq T_1$, the M_i-saturations operate in their linear region: for all $t \geq T_1$ the trajectory satisfies the equations (21). ■

The *delay-dependent* stability analysis of the linear system (21) follows. The details of the proof are given in Appendix A.

Lemma 6 *The characteristic quasipolynomial of the system*

$$\dot{x}_i = \sum_{j=i+1}^{n} x_j - \sum_{j=1}^{n} \bar{x}_j, i = 1, ..., n \tag{24}$$

is

$$p_n(s, e^{-\tau s}) = (s+1)^n + (e^{-\tau s} - 1)[(s+1)^n - s^n] \ . \tag{25}$$

Moreover, this quasipolynomial is stable for all $\tau \in [0, \tau_n^*]$ *with*

$$\tau_n^* = \frac{1}{3n} \ . \tag{26}$$

Remark 5 *The value* τ_n^* *is a lower bound for the maximal allowed delay* τ_n^e *for which the system (24) is stable. To determine the exact bound, one has to find the solutions of a polynomial equation of degree* $2n$ *which characterizes the existence of at least one (non-zero) root of the corresponding characteristic equation on the imaginary axis. When* $n = 1, 2$*, such a study can be carried out and leads to the exact bounds* $\tau_1^e = \frac{\pi}{2}$ *and* $\tau_2^e = \frac{1}{\sqrt{2+\sqrt{5}}} \arctan(2\sqrt{2+\sqrt{5}})$*. Moreover one can prove that in each case, the corresponding linear system is stable for any delay* $\tau \in [0, \tau_n^e), n = 1, 2$ *and unstable for all* $\tau > \tau_n^e$*. For* n *greater than* 3 *the computations are quite involved.*

The above lemmas allow us to establish Theorem 4 in a straightforward manner.

Proof. (Theorem 4)

According to Lemma 5, for any trajectory of the system (6), (7) in closed-loop with the control (19) there exists an instant $T_1 \geq 0$ such that when $0 \leq \tau \leq \breve{\tau}_n$ with $\breve{\tau}_n$ given by (20) then for all $t \geq T_1$, the trajectory is within the region of the space where all the M_i-saturations operate in their linear region, which implies that, on $[T_1, +\infty[$, the trajectory is solution of the linear system (24).

As stated in Lemma 6, the system (24) is asymptotically stable for all $\tau \in [0, \tau_n^*)$ where $\tau_n^* > 0$ is given by (26). We conclude that the control law (18) globally uniformly asymptotically stabilizes system (6), (7) for all $\tau \in [0, \widehat{\tau})$ with $\widehat{\tau} = \min\{\tau_n^*, \breve{\tau}_n\}$. ■

3.3 Stability for arbitrarily large delays

Theorem 4 can be applied to the system (6), (7) when the delay is sufficiently small. It follows almost straightforwardly from this theorem, Proposition 2 and Lemma 1 that the chain of integrators (1), (2) with an arbitrarily large delay can also be globally uniformly asymptotically stabilized by bounded feedback. It turns out that one can prove the following stronger result:

Corollary 7 *For all $\tau \geq 0$, the system (1), (2) is globally uniformly asymptotically stabilizable by arbitrarily small feedbacks.*

Proof. The result is well-known when $\tau = 0$. Consider the system (1), (2) with delay in the input $\tau > 0$. According to Theorem 4 and Proposition 2, there exists a control (18) bounded in norm by M_n where the M_i-saturations $\sigma_i(\cdot)$ are such that $M_i \leq \frac{1}{8} M_{i+1}, i = 1, \ldots, n-1$ that stabilizes the system

$$w_i(t) = w_{i+1}(t), i = 1, \ldots, n-1 \ ,$$

$$w_n(t) = u(t - \widehat{\tau}) \ ,$$

when $0 < \hat{\tau} \leq \min\{\tau_n^*, \breve{\tau}_n\}$. A possible choice for the constants M_i is $M_i = 8^{i-1}M_1$ for all $i \geq 1$. Such a choice results in the following requirement on $\hat{\tau}$:

$$0 \leq \hat{\tau} \leq \min\left\{\frac{1}{3n}, \frac{1}{9}, \frac{7}{8n + 4.8^n - 32}\right\} . \tag{27}$$

Observe that the right hand side of this inequality is independent from M_1. This fact will be exploited to prove that some feedbacks arbitrarily small in norm are stabilizing.

The expression of stabilizing feedbacks is, according to Proposition 2,

$$u_{Sw}(t-\hat{\tau}) = -\sigma_n(h_n(w(t-\hat{\tau})) + \sigma_{n-1}(h_{n-1}(w(t-\hat{\tau})) + \ldots + \sigma_2(h_2(w(t-\hat{\tau})) + \sigma_1(h_1(w(t-\hat{\tau}))) \ldots))$$

where $h_i(w(t-\hat{\tau})), i = 1, ..., n$ are defined in Proposition 2.

Finally, it follows from Lemma 1 that the system (1), (2) with delay in the input τ is stabilized by the control law

$$u(t-\tau) = -\left(\frac{\hat{\tau}}{\tau}\right)^n \sigma_n(f_n(\bar{y}) + \sigma_{n-1}(f_{n-1}(\bar{y}) + \ldots + \sigma_2(f_2(\bar{y}) + \sigma_1(f_1(\bar{y}))) \ldots)) \tag{28}$$

where

$$f_{n-i}(y) := \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} \left(\frac{\hat{\tau}}{\tau}\right)^{-n+j+1} y_{n-j} , i = 0, ..., n-1 .$$

Let ε be a strictly positive number. Since no restriction is imposed on M_1, and in particular there is no relation between M_1, $\hat{\tau}$ and τ, a possible choice for M_1 is

$$M_1 = \frac{\tau^n}{8^{n-1}\left(\min\left\{\frac{1}{3n}, \frac{1}{9}, \frac{7}{8n+4.8^n-32}\right\}\right)^n}\varepsilon . \tag{29}$$

The feedback (28) is bounded in norm by $8^{n-1}\left(\frac{\hat{\tau}}{\tau}\right)^n M_1$. According to (27) and (29), it follows that this feedback is bounded by ε. So, the largest value of the norm of the feedback (28) can be chosen arbitrarily small. ■

Remark 6 *An important consequence of the above corollary is that a design robust with respect to delay uncertainties can be achieved. Indeed, the design of the controller based on a conservative upper bound of the time delay will ensure stability for smaller delays.*

4 Third-order example

To validate the control law designs we have proposed, we study by means of simulations the behavior of the trajectories of the three dimensional chain of integrators in closed-loop with the feedbacks deduced from Theorem 4 and Corollary 7.

Numerical computations provide an approximation of τ_3^e, the largest possible delay such that the linear system (24) when $n = 3$ is asymptotically stable: the value obtained is $\tau_3^e = 0.405$. Furthermore, for all delays $\tau > \tau_3^e$ the system is unstable.

First, we focus our attention on the case of a small delay. It turns out that the bound τ_3^e seems to be crucial in practice: indeed, although the bound $\hat{\tau}$ given in the proof of Theorem 4 is much smaller, according to simulations, the stabilization scheme works properly as soon as the delay τ is smaller than τ_3^e. We illustrate this fact in Figure 1 with $\tau = 0.2$ and $M_1 = 0.2$.

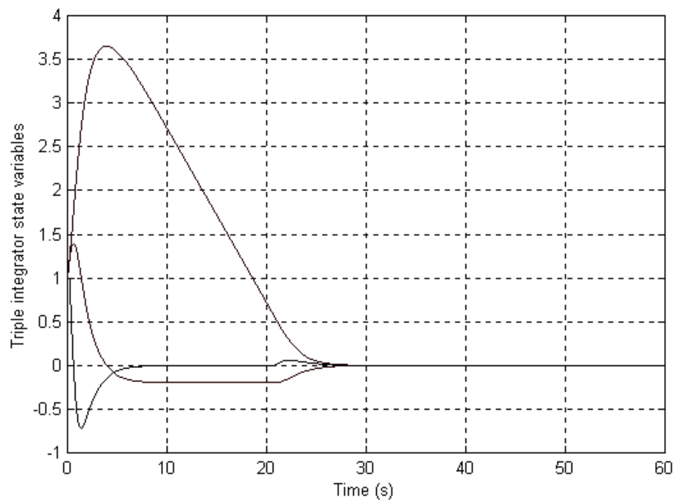


Figure 1: Forwarding scheme for $\tau = 0.2s$

As expected, if the delay is greater than the bound τ_3^e, the closed-loop system is not asymptotically stable. In fact, this system is not even locally asymptotically stable because the stability of the linear system which approximates it at the origin is not asymptotically stable. We illustrate this phenomenon in Figure 2 when $\tau = 0.6$, with the same choice for the saturations.

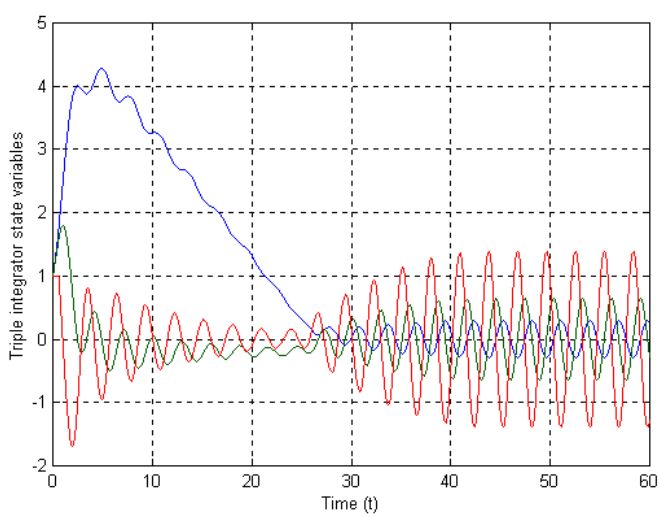
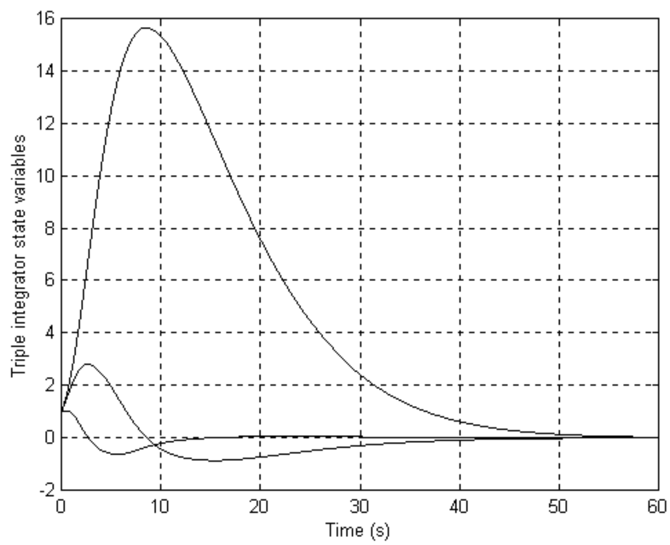


Figure 2: Forwarding scheme for τ =0.6s

Next, we consider the case of a large delay: we choose $\tau = 1.0$. We observe on Figure 3 that in this case the control law (28) (which is obtained via the homogeneous transformation of Lemma 1) globally uniformly asymptotically stabilizes the system. In this case the choices $M_1 = 20$ and $\frac{\hat{\tau}}{\tau} = 0.2$ were made. Notice that the factor $\left(\frac{\hat{\tau}}{\tau}\right)^n$ in the expression of the stabilizing control laws (28) allows to choose larger values for M_1 resulting in a control law bounded by $8^2 M_1 \left(\frac{\hat{\tau}}{\tau}\right)^3$.



.

Figure 3: Forwarding scheme with homogeneous transformation, $\tau = 1s$

Finally, the tuning of the parameters shows the trade-off between response and saturation level which is inherent to the nature of these control laws.

5 Concluding remarks

We have restricted our attention to the problem of stabilizing by bounded feedback a chain of integrators of dimension n with an arbitrary delay in the input. We conjecture that the main result of our work can be extended to larger classes of systems and in particular to linear null-controllable systems and the family of nonlinear systems termed strict feedforward systems.

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A Appendix: proof of Lemma 6

Proof. To simplify, we denote $z := e^{-\tau s}$. The characteristic quasipolynomial of the system (24) is

$$p_n(s,z) = \det \begin{pmatrix} s+z & z-1 & z-1 & \cdots & \cdots & z-1 \\ z & s+z & z-1 & \cdots & \cdots & z-1 \\ z & z & \ddots & \ddots & & z-1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & s+z & z-1 \\ z & z & z & \cdots & z & s+z \end{pmatrix} .$$

Lengthy but simple calculations lead to

$$p_n(s,z) = (s+1)^n + (z-1)[(s+1)^n - s^n] .$$

A $\mathcal{D}$-partition analysis (see [7]) is performed to study the stability of this quasipolynomial. It is based on the fact that the quasipolynomial (25) is stable when $\tau = 0$ and that the location of the roots changes continuously with respect to τ. Therefore, as τ grows the system becomes unstable only when a root, or a pair of roots hit for the first time the imaginary axis. It follows that to determine the interval $[0, \tau_n^e)$ such that the system (24) is asymptotically stable when τ belongs to this interval, we need to find τ_n^e, the smallest value of τ for which the equation

$$p_n(j\omega, e^{-j\omega\tau}) \; = \; 0 \tag{30}$$

admits a solution $\omega \in R$. To overcome the difficult problem of finding for any n explicit expressions of τ_n^e, we will simply exhibit a sequence of real numbers $\tau_n^* > 0$ such that for all $\tau \in [0, \tau_n^*]$, the equation (30) admits no solution ω. Obviously, $\tau_n^* \leq \tau_n^e$.

Notice first that (30) implies

$$e^{-j\omega\tau}(j\omega + 1)^n - (e^{-j\omega\tau} - 1)(j\omega)^n \; = \; 0 \; .$$

This equality is not satisfied when $\omega = 0$. When $\omega \neq 0$, it is equivalent to the equality

$$e^{j\omega\tau} \; = \; 1 - \left(1 - j\omega^{-1}\right)^n \; . \tag{31}$$

We distinguish now between two cases.

First case: $|\omega| \geq 3n$.

$>$From (31) we deduce that

$$|1 - \left(1 - j\omega^{-1}\right)^n | \; = \; 1 \; . \tag{32}$$

On the other hand,

$$
\begin{aligned}
\left|1-\left(1-j\omega^{-1}\right)^{n}\right| &\leq \left|\sum_{p=1}^{n}\frac{n!}{p!(n-p)!}\left(-j\omega^{-1}\right)^{p}\right| \\
&\leq \sum_{p=1}^{n}\frac{n!}{p!(n-p)!}\frac{1}{(3n)^{p}} \\
&\leq \sum_{p=1}^{n}\frac{1}{3^{p}p!} \leq \frac{1}{2}\ .
\end{aligned}
\tag{33}
$$

It follows that (32) does not admit any solution ω such that $|\omega| \geq 3n$, which implies that (31) does not admit any solution ω such that $|\omega| \geq 3n$.

Second case: $0 < |\omega| \leq 3n$.

$>$From (31) we deduce that

$$
\left|1-e^{j\omega\tau}\right| = \left|\left(1-j\omega^{-1}\right)^{n}\right| = \left(1+\frac{1}{\omega^{2}}\right)^{\frac{n}{2}} \geq 1\ . \tag{34}
$$

On the other hand, one can check readily that

$$
\left|1-e^{j\omega\tau}\right| = \sqrt{2}\left|\sin\left(\frac{1}{2}\omega\tau\right)\right| \leq \frac{\sqrt{2}}{2}|\omega\tau|\ . \tag{35}
$$

Combining (34) and (35), the inequality

$$
|\omega\tau| \geq \sqrt{2} \tag{36}
$$

is obtained. When $\tau \in [0, \tau_n^*]$, this inequality and $|\omega| \leq 3n$ imply that

$$
|3n\tau_n^*| \geq \sqrt{2}\ . \tag{37}
$$

According to the definition of τ_n^* defined in (26), this inequality is not satisfied. It follows that (32) does not admit any solution ω such that $|\omega| \leq 3n$, which implies that (31) does not admit any solution ω such that $|\omega| \leq 3n$. This concludes the proof. ■

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