Explicit Construction of Viability Kernels for Sustainable Management of Ecosystems with an Application to the Hake–Anchovy Peruvian Fisheries

Eladio OCAÑA [1], Michel DE LARA[2], Ricardo OLIVEROS–RAMOS [3] and Jorge TAM [3]

9th June 2008

[1]IMCA-FC, Universidad Nacional de Ingeniería, Lima–Perú
[2]CERMICS, Université Paris-Est, France
[3]Instituto del Mar del Perú, Centro de Investigaciones en Modelado Oceanográfico y Biológico Pesquero (CIMOBP), Callao–Perú

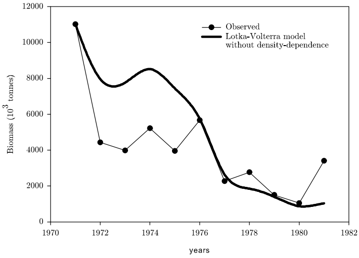
Lotka–Volterra model without density–dependence

As an illustration of a Lotka–Volterra system, let us consider the following equations in discrete time,

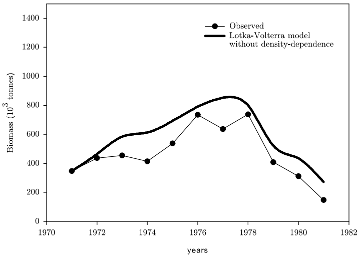
$$\begin{cases} x(t+1) &= Rx(t) - \alpha x(t)y(t) - u(t)x(t) \;, \\ y(t+1) &= Ly(t) + \beta y(t)x(t) - v(t)y(t) \;, \end{cases}$$

where $R > 1$, $0 < L < 1$, $\alpha > 0$ and $\beta > 0$.

Hake–anchovy Peruvian fisheries between 1971 and 1981



(a) Anchovy



(b) Hake

Figure: Comparison of observed and simulated biomasses of anchovy and hake using a Lotka–Volterra model without density-dependence. Model parameters are $R = 1.980$, $L = 0.955$, $\alpha = 1.22 \times 10^{-6}\ t^{-1}$, $\beta = 4.15 \times 10^{-8}\ t^{-1}$.

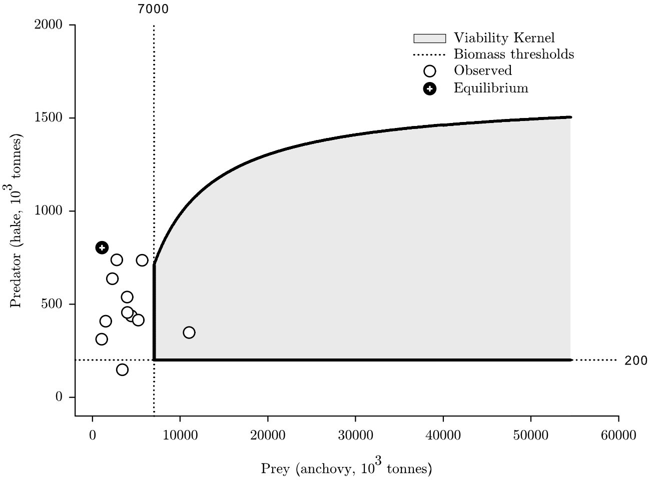


Figure: Viability kernel for a Lotka–Volterra model without density–dependence, in the predator–prey phase space

Outline of the presentation

DISCRETE TIME VIABILITY

Condiser the **control system**

$$\begin{cases} x(t+1) = f(x(t), u(t)) \text{ for all } t \in \mathbb{N}, \\ x(0) = x_0 \quad \text{given,} \end{cases}$$

where

- **time** $t \in \mathbb{N}$ is discrete (*the time period* $[t, t+1[$ *may be a year, a month, etc.*),
- the **state variable** $x(t)$ belongs to the finite dimensional state space $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$ (*biomasses, abundances, etc.*),
- the **control variable** $u(t)$ is an element of the control set $\mathbb{U} = \mathbb{R}^{n_{\mathbb{U}}}$ (*catches or harvesting effort*),
- the **dynamics** f maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$ (*age-class model, population dynamics, etc.*).

Desirable set

A controller or a decision maker describes "desirable configurations of the system" through a set $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ termed the **desirable set**

$$\big(x(t), u(t)\big) \in \mathbb{D} \text{ for all } t \in \mathbb{N} ,$$

where $\mathbb{D}$ includes both system states and controls constraints.

Typical instances of such a desirable set are given by inequalities requirements:
$\mathbb{D} = \{(x, u) \in \mathbb{X} \times \mathbb{U} \mid \forall i = 1, \ldots, p , \quad g_i(x, u) \geq 0\}.$

The desirable set $\mathbb{D}$ may capture **conflicting requirements, such as biological and economical**.

Viability kernel

The set

$$\mathbb{V}(f,\mathbb{D}) := \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \exists\ (u(0),u(1),\ldots) \text{ and } (x(0),x(1),\ldots) \\ \quad \text{satisfying} \\ x(t+1) = f(x(t),u(t)) \text{ for all } t \in \mathbb{N} \\ \quad \text{and } (x(t),u(t)) \in \mathbb{D} \text{ for all } t \in \mathbb{N} \end{array} \right\}$$

is called the **viability kernel** associated with the dynamics f and the desirable set $\mathbb{D}$.

From an initial point $x_0 \in \mathbb{V}(f,\mathbb{D})$ can start a trajectory which satisfies dynamics and constraints.

The **viability kernel until time** k **associated with** f **in** $\mathbb{D}$ is

$$\mathbb{V}_k := \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \exists \, (u(0), u(1), \ldots, u(k)) \text{ and } (x(0), x(1), \ldots, x(k)) \\ \quad \text{satisfying } x(t+1) = f(x(t), u(t)) \\ \quad \text{for } t = 0, \ldots, k-1 \\ \quad \text{and} \\ (x(t), u(t)) \in \mathbb{D} \text{ for } t = 0, \ldots, k \end{array} \right\}$$

Dynamic programming equation

$$\mathbb{V}_{k+1} = \{x \in \mathbb{V}_k \mid \exists u \in \mathbb{U}\,,\ (x,u) \in \mathbb{D} \text{ and } f(x,u) \in \mathbb{V}_k\}\ .$$

Such an algorithm provides approximation from above of the viability kernel as follows:

$$\mathbb{V}(f,\mathbb{D}) \subset \bigcap_{k \in \mathbb{N}} \mathbb{V}_k = \lim_{k \to +\infty} \downarrow \mathbb{V}_k\ .$$

Proposition

The following relations are equivalent.

1. $\mathbb{V}_{k+1} = \mathbb{V}_k$.
2. $\mathbb{V}_k$ *is a viability domain of* f *in* $\mathbb{D}$.
3. $\mathbb{V}_k = \mathbb{V}(f, \mathbb{D})$.

VIABLE CONTROL OF GENERIC NONLINEAR ECOSYSTEM MODELS

Generic nonlinear ecosystem models

For simplicity, we consider a two–dimensional state model

$$\begin{cases} x(t+1) = x(t)g(x(t), y(t), u(t)) , \\ y(t+1) = y(t)h(x(t), y(t), v(t)) , \end{cases}$$

where

- state vector (x, y) represents **biomasses**,
- control vector (u, v) is **fishing effort** of each species, respectively.

The **catches** are thus ux and vy (measured in biomass).

Conservation and minimal catch

The desirable set

$$\mathbb{D} = \{ (x, y, u, v) \in \mathbb{R}^4 \mid x \geq x^\flat, \; y \geq y^\flat, \; ux \geq X^\flat, \; vy \geq Y^\flat \} \, .$$

$\mathbb{D}$ is defined by

- **minimal biomass thresholds** $x^\flat \geq 0$, $y^\flat \geq 0$,
- **minimal catch thresholds** $X^\flat \geq 0$, $Y^\flat \geq 0$.

Proposition

Assume that

- *the function* $g : \mathbb{R}^3 \to \mathbb{R}$ *is continuously decreasing in the control* u *and satisfies* $\lim_{u \to +\infty} g(x, y, u) \leq 0$,
- *the function* $h : \mathbb{R}^3 \to \mathbb{R}$ *is continuously decreasing in the control variable* v, *and satisfies* $\lim_{v \to +\infty} h(x, y, v) \leq 0$.

If the thresholds are such that

$$g(x^\flat, y^\flat, \frac{X^\flat}{x^\flat}) \geq 1 \ \text{ and } \ h(x^\flat, y^\flat, \frac{Y^\flat}{y^\flat}) \geq 1 \, ,$$

the viability kernel is given by $\mathbb{V}(f, \mathbb{D}) =$

$$\left\{ (x, y) \mid x \geq x^\flat, y \geq y^\flat, xg(x, y, \frac{X^\flat}{x}) \geq x^\flat, yh(x, y, \frac{Y^\flat}{y}) \geq y^\flat \right\} .$$

Adjusting catches to prominent biomass conservation thresholds

Considering that minimal biomass conservation thresholds $x^\flat \geq 0$, $y^\flat \geq 0$ are given first (for prominent biological issues), we shall now examine conditions for the existence of minimal catch thresholds $X^\flat \geq 0$, $Y^\flat \geq 0$ susceptible to be sustainably maintained.

Proposition

Suppose that the assumptions of Proposition 2 are satisfied. A necessary and sufficient condition for the existence of nonnegative minimal catch thresholds $X^\flat \geq 0$, $Y^\flat \geq 0$ *is*

$$g(x^\flat, y^\flat, 0) \geq 1 \text{ and } h(x^\flat, y^\flat, 0) \geq 1 .$$

In this case, the following **maximum sustainable catches**

$$\begin{cases} X^{\flat,\star} & := \quad x^\flat \max\{u \geq 0 \mid g(x^\flat, y^\flat, u) \geq 1\} \\ Y^{\flat,\star} & := \quad y^\flat \max\{v \geq 0 \mid h(x^\flat, y^\flat, v) \geq 1\} \end{cases}$$

are nonnegative, and any couple $(X^\flat, Y^\flat) \in [0, X^{\flat,\star}] \times [0, Y^{\flat,\star}]$ *is such that* $X^\flat$ *and* $Y^\flat$ *are levels of production which can be sustainably maintained.*

VIABILITY ANALYSIS OF THE HAKE–ANCHOVY PERUVIAN FISHERIES

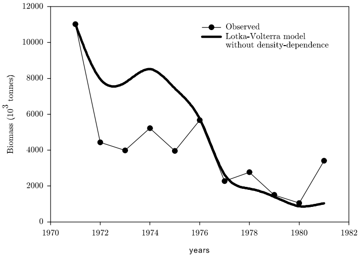
Lotka–Volterra model without density–dependence

As an illustration of a Lotka–Volterra system, let us consider the following equations in discrete time,

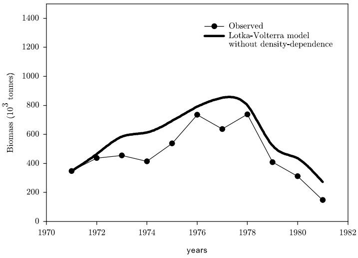
$$\begin{cases} x(t+1) &= Rx(t) - \alpha x(t)y(t) - u(t)x(t) \,, \\ y(t+1) &= Ly(t) + \beta y(t)x(t) - v(t)y(t) \,, \end{cases}$$

where $R > 1$, $0 < L < 1$, $\alpha > 0$ and $\beta > 0$.

Hake–anchovy Peruvian fisheries between 1971 and 1981



(a) Anchovy



(b) Hake

Figure: Comparison of observed and simulated biomasses of anchovy and hake using a Lotka–Volterra model without density-dependence. Model parameters are $R = 1.980$, $L = 0.955$, $\alpha = 1.22 \times 10^{-6}\ t^{-1}$, $\beta = 4.15 \times 10^{-8}\ t^{-1}$.

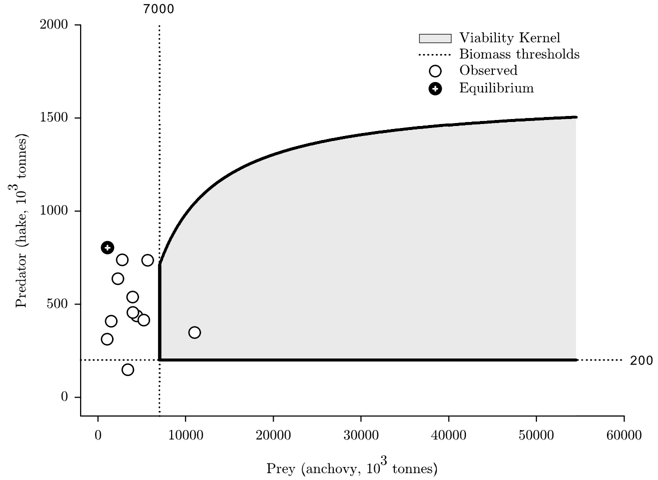


Figure: Viability kernel for a Lotka–Volterra model without density–dependence, in the predator–prey phase space

The predator–prey plane can be divided in 5 domains:

1. inside the viability kernel with biomasses that can be sustainably harvested,
2. $\{(x,y)|x > x^\flat, y < y^\flat\}$, with non viable biomasses due to predator scarcity,
3. $\{(x,y)|x < x^\flat, y > y^\flat\}$, with non viable biomasses due to prey scarcity,
4. $\{(x,y)|x > x^\flat, y > y^\flat\,(x,y) \notin \mathbb{V}(f,\mathbb{D})\}$, with non viable biomasses due to high predator pressure, and
5. $\{(x,y)|x < x^\flat, y < y^\flat\}$, with non viable biomasses due to both predator and prey scarcity.

Most observed biomasses lie outside the viability kernel between 1971 and 1981 (Figure 4). Specifically, they were located in the third domain indicating a scarcity of anchovy, in agreement to its post–collapse recovery situation.

Some points were located in the fourth domain indicating both predator and prey scarcity, acording to post–colapse situation of anchovy and overfishing of hake at the end of 1970s.

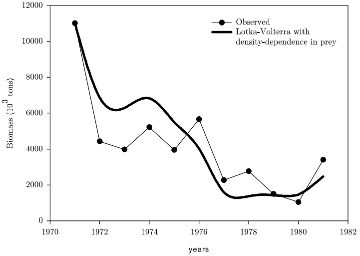
These results suggest that, during this period, more conservative measures would have been required to ensure the viability of both species, and a faster recovery of anchovy.

Lotka–Volterra model with density–dependence in the prey

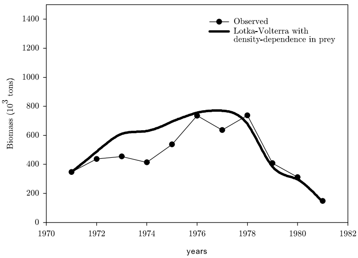
$$\begin{cases} x(t+1) & = & Rx(t) - \frac{R}{\kappa}x(t)^2 - \alpha x(t)y(t) - u(t)x(t) \, , \\ y(t+1) & = & Ly(t) + \beta y(t)x(t) - v(t)y(t) \, , \end{cases}$$

where $R > 1$, $0 < L < 1$, $\alpha > 0$, $\beta > 0$ and $\kappa = \frac{R}{R-1}K$, with K *the* carrying capacity for prey.

Hake–anchovy Peruvian fisheries between 1971 and 1981



(a) Anchovy



(b) Hake

Figure: Comparison of observed and simulated biomasses of anchovy and hake using a Lotka–Volterra model with density-dependence in the prey. Model parameters are $R = 2.24$, $L = 0.98$, $\kappa = 64\,672 \;\times 10^3$ t ($K = 35\,800 \;\times 10^3$ t), $\alpha = 1.230 \times 10^{-6}\; t^{-1}$, $\beta = 4.326 \times 10^{-8}\; t^{-1}$.

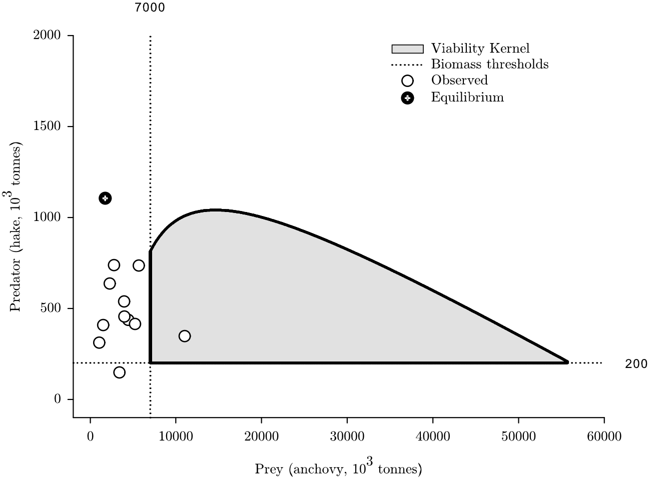


Figure: Viability kernel for a Lotka–Volterra model with density-dependence in the prey in the predator–prey phase space