Viable harvest of monotone bioeconomics models

Production issues with applications to fishery management

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Outline

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Age structured model

- the state : $N = (N_a)_{a=1,\dots,A} \in \mathbb{R}_+^A$, the *abundances* at age
- the control : λ the *fishing effort multiplier*
- the dynamics : $N(t+1) = g(N(t), \lambda(t))$ given by

$$\begin{cases} g_1(N,\lambda) & = & \varphi(SSB(N)) \,, \\ g_a(N,\lambda) & = & e^{-(M_{a-1}+\lambda F_{a-1})} N_{a-1} \,, \quad a = 2, \dots, A-1 \,, \\ g_A(N,\lambda) & = & e^{-(M_{A-1}+\lambda F_{A-1})} N_{A-1} + e^{-(M_A+\lambda F_A)} N_A \,. \end{cases}$$

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where

- the *spawning stock biomass* SSB is defined by

$$SSB(N) := \sum_{a=1}^{A} \gamma_a w_a N_a$$

- the function φ describes the *stock-recruitment relationship*
- M_a is the natural *mortality rate* of individuals of age a
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Harvested fish population age structured model

Example

Typical stock-recruitment relationship:

- *Constant:* $\varphi(B) = R$.
- *Linear:* $\varphi(B) = RB$.
- *Beverton-Holt:* $\varphi(B) = \frac{B}{\alpha+\beta B}$.
- *Ricker:* $\varphi(B) = \alpha B e^{-\beta B}$.

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The harvest term

The exploitation is described by catch-at-age C_a and yield Y, both defined for a given vector of abundance N and a given control λ :

$$C_a(N,\lambda) = \frac{\lambda F_a}{\lambda F_a + M_a}\left(1 - e^{-(M_a+\lambda F_a)}\right) N_a$$

The production in term of biomass is :

$$Y(N,\lambda) = \sum_{a=1}^{A} w_a\, C_a(N,\lambda)$$

Questions

- Given a desirable level of landings (tons), what are the vectors of abundances $N = (N_a)_{a=1,..,A}$ (initial conditions) for which one can always harvest at least that quantity?
- What levels of catch (landings) are non sustainable?
- Given an abundance at age $N = (N_a)_{a=1,..,A}$ What is the maximum sustainable yield starting from N respecting preservation constraints?

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Discrete time control system

Let us consider a nonlinear control system described in discrete time by the difference equation

$$\begin{cases} N(t+1) = g(N(t), \lambda(t)), & t \geq t_0, \\ N(t_0) = N_0 \quad \text{given}, \end{cases}$$

where

- The state variable $N(t)$ belongs to the state space $\mathbb{X} \subseteq R^{n_\mathbb{X}}$.
- The control variable $\lambda(t)$ is an element of the control set $\mathbb{U} \subseteq R^{n_\mathbb{U}}$.
- The dynamics g maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$.

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Desirable configurations

A decision maker describes desirable configurations of the system through a set $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ termed the desirable set

$$(N(t), \lambda(t)) \in \mathbb{D}, \quad t \geq t_0,$$

where $\mathbb{D}$ includes both system states and controls constraints.

Example

- $\mathbb{D}_{\mathrm{protect}} := \{(N, \lambda) \ : \ N \geq \bar{N}\}$
- $\mathbb{D}_{y_{\min}} := \{(N, \lambda) \ : \ Y(N, \lambda) \geq y_{\min}, \quad SSB(N) \geq B_{\lim}\}$
- $\mathbb{D}_{\mathrm{ICES}} := \{(N, \lambda) \ : \ SSB(N) \geq B_{\lim}\}$

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Viability kernel

Definition

- *Viability kernel*

$$\mathbb{V}(g, \mathbb{D}) = \left\{ \begin{array}{l} N_0 \in \mathbb{X} \ : \ \textit{there exist} \ \ \lambda(t_0), \lambda(t_0+1), \ldots \\ \ N(t_0), N(t_0+1), \ldots \textit{such that } N(t_0) = N_0 \\ N(t+1) = g(N(t), \lambda(t)) \ \ \textit{and} \\ (N(t), \lambda(t)) \in \mathbb{D} \end{array} \right.$$

Goal

- *Determine or approximate the viability kernel* $\mathbb{V}(g, \mathbb{D})$ *for a given dynamics* g *and a given desirable set* $\mathbb{D}$*. domain.*

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Monotonicity properties on the dynamics

Definition

We say that the dynamics $g : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ *is a monotone bioeconomic dynamics if g is increasing with respect to the state i.e.*

$$N' \geq N \Rightarrow g(N', \lambda) \geq g(N, \lambda)$$

and is decreasing with respect to the control i.e.

$$\lambda' \geq \lambda \Rightarrow g(N, \lambda') \leq g(N, \lambda)$$

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Production desirable sets

Definition

A desirable set $\mathbb{D}$ is said to be a production desirable set if $\mathbb{D}$ is increasing w.r.t. both the state and to the control, that is

$$\textit{if } (N,\lambda) \in \mathbb{D} \textit{ then } (N',\lambda') \in \mathbb{D} \quad \textit{for all } N' \geq N,\ \lambda' \geq \lambda$$

Example

$$\mathbb{D}_{y_{\min}} := \{(N,\lambda) \ : \ Y(N,\lambda) \geq y_{\min}, \quad SSB(N) \geq B_{\lim}\},$$

where $Y : \mathbb{X} \times \mathbb{U} \longrightarrow R$ is increasing w.r.t. both variables (state and control).

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Viability kernel estimates

Assume that $\lambda_\flat \leq \lambda \leq \lambda_\sharp$ for all $\lambda \in \mathbb{U}$.

Proposition

If we consider the dynamics $g_\flat(\cdot) = g(\cdot, \lambda_\flat)$ *and* $g_\sharp(\cdot) = g(\cdot, \lambda_\sharp)$ *then, for a production desirable set* $\mathbb{D}$ *we have*

- $$((g_\sharp)^t(N), \lambda_\flat) \in \mathbb{D} \quad t \geq t_0 \;\Rightarrow\; N \in \mathbb{V}(g, \mathbb{D})$$

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Maximum sustainable yield

Assume the existence of a steady state $\overline{N}(\lambda)$ for the dynamics $N \mapsto g(N, \lambda)$, for all $\lambda \in [\lambda_\flat, \lambda_\sharp]$.
Given a yield function $Y : \mathbb{X} \times \mathbb{U} \longrightarrow \boldsymbol{R}$, we define the *maximum sustainable yield* by

$$MSY = \sup_{\lambda \in [\lambda_\flat, \lambda_\sharp]} Y(\overline{N}(\lambda), \lambda)$$

Maximum sustainable yield

Consider the production desirable set $\mathbb{D}_{y_{\min}}$ given by

$$\mathbb{D}_{y_{\min}} = \{(N, \lambda) \mid Y(N, \lambda) \geq y_{\min}\} .$$

Proposition

Suppose that the yield function $Y : \mathbb{X} \times \mathbb{U} \longrightarrow \boldsymbol{R}$ is increasing with respect both to the state and to the control. Then,

- $$MSY \geq y_{\min} \Rightarrow \mathbb{V}(g, \mathbb{D}_{y_{\min}}) \neq \emptyset .$$
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Minimal viable feedback

Given a desirable level of production y_{min}, for some states N we can define the feedback control $\lambda^{\star}_{y_{\text{min}}}(N) \in [\lambda_{\flat}, \lambda_{\sharp}]$ where

$$Y(N, \lambda^{\star}_{y_{\text{min}}}(N)) = y_{\text{min}}$$

Proposition

If we consider the dynamics

$$g^{\star}_{y_{\text{min}}}(N) = g(N, \lambda^{\star}_{y_{\text{min}}}(N))$$

then,

$$\mathbb{V}(g, \mathbb{D}_{y_{\text{min}}}) = \mathbb{V}(g^{\star}_{y_{\text{min}}}, \mathbb{D}_{y_{\text{min}}})$$

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- F_a is the mortality rate of individuals of age a due to harvesting

The harvest term

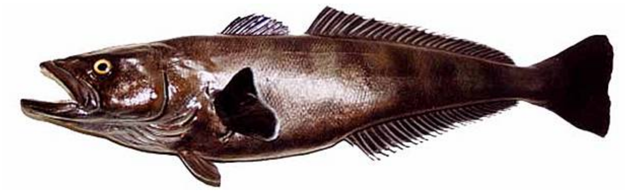
The exploitation is described by catch-at-age C_a and yield Y, both defined for a given vector of abundance N and a given control λ :

$$C_a(N,\lambda) = \frac{\lambda F_a}{\lambda F_a + M_a}\left(1 - e^{-(M_a+\lambda F_a)}\right) N_a$$

The production in term of biomass is :

$$Y(N,\lambda) = \sum_{a=1}^{A} w_a\, C_a(N,\lambda)$$

The Patagonian toothfish (Légine australe)[1]



- Abundance at age (state): $N = (N_a)_{a=1,..,A}$
  Patagonian toothfish $A = 36$
- Fishing effort multiplier (control): $\lambda \in \mathbb{U} = [\lambda^\flat, \lambda^\sharp]$
  Patagonian toothfish $\lambda^\flat = 0, \quad \lambda^\sharp = 0.3$
- Stock-recruitment relationship φ
  Patagonian toothfish

$$\varphi(B) = \frac{B}{\alpha + \beta B}$$

[1] Data: CEPES, SUBPESCA, Chile

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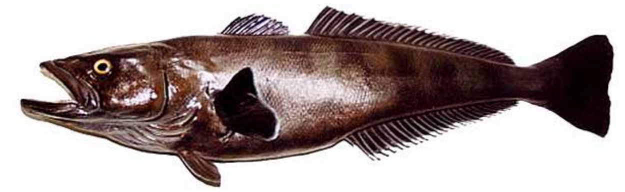

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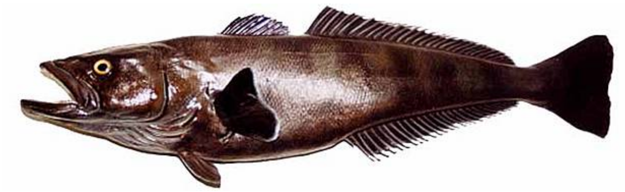

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Some questions about sustainability of landings

- Given a desirable level of landings (tons), what are the vectors of abundances $N = (N_a)_{a=1,..,A}$ (initial conditions) for which one can always harvest at least that quantity?
- For a desirable level of landing $y_{\min}$ we have to compute $\mathbb{V}(g, \mathbb{D}_{y_{\min}})$ where

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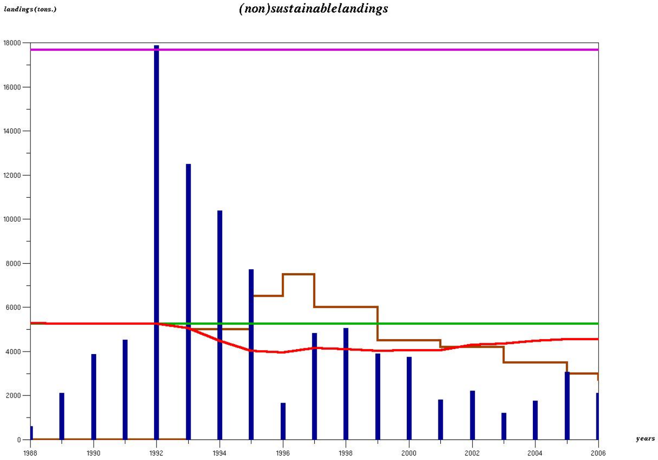
Results

■	=	landings
red line	=	maximum sustainable yield (considering the abundance)
dark red line	=	quota of the government regulatory Chilean agency (SUBPESCA)
magenta line	=	non sustainable level (independently of the abundance)
green line	=	maximum sustainable yield in the equilibrium



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