Using participating and financial contracts to insure catastrophe risk: Implications for crop risk management

Geoffroy Enjolras *

0. Abstract

High losses generated by natural catastrophes reduce the availability of insurance. Among the ways to secure risk, the subscription of participating and non-participating contracts respectively permit to implement the two major principles in risk allocation: the mutuality and the transfer principles. Decomposing a global risk in its idiosyncratic and systemic components, we show that, combined, the participating contracts cover against the individual losses under a variable premium and filters the systemic risk, which is covered with a non-participating contract under a fixed premium. Reunifying and improving Doherty and Schlesinger (2002) and Mahul (2002) approaches, we replace the non-participating contract by a financial one based on an index straightly correlated to the systemic risk, under a basis risk. Then, we prove that the combination of both participating and financial contracts offers an unbiased coverage that eliminates the basis risk and provides a sustainable solution for the insurer and the stakeholder. Therefore, potential applications for crop risk management are studied.

Keywords: Catastrophe risk, Crop insurance, Optimal hedging, Participating insurance policy, Securitization

JEL Number: D81, G22, Q14

1. Introduction

In recent years, more and more developed countries have modernized the insurance against natural events by redistributing the roles of the main actors. Henceforth, the states play a paramount role when private solutions are not forthcoming. However, there exist several limits to the ability of private insurance and reinsurance to fund catastrophic losses. The two main critics are upon the financial reserve the insurers must cup with (Jaffee and Russell, 1997) and agency conflicts (Froot, 2000). Our model permits to reduce these two sources of inefficiency.

Marshall (1974) noticed that a mechanism of risk-sharing exists in which individuals are direct actors and that permit an allocation of collective risk: mutual participating contract: "Mutual and participating stock insurance companies issue contracts which include, besides the obligation to indemnify loss, a dividend to the consumer which depends on the overall performance of the company" (pp. 483). Participating insurance policy is a policy in which the insured completely cover the idiosyncratic component of his individual risk but he receives a dividend or respectively pays an extra premium, if the aggregation of the insurers' contracts is profitable, respectively insufficient. Although individuals covered by this kind of contract gain the same "dividend", and not a dividend proportional to their risk tolerance, this kind of mechanism seems to be able to procure a more efficient repartition of the risk than a single

* PhD Student and Teaching Assistant, University of Montpellier 1
  LAMETA - INRA, 2 place Pierre Viala, 34060 Montpellier Cedex 1, France
  Mail : enjolras@ensam.inra.fr
insurance contract that only permits to diversify individual risk. This hypothesis is explored and validated by Dionne and Doherty (1993). Their model includes both individual and social (or systemic) risk. Thus, insurers can propose contracts determined according to the individual risk only or that include a dividend conditioned to the realization of a particular social state. In this case, resource allocation is Pareto-superior.

In his famous article, Raviv (1979) demonstrates that when losses are correlated, which is typically the case during a natural event, the optimal design of an insurance contract is based on a risk decomposition in two elements: a systemic risk (not diversifiable) and an individual or idiosyncratic risk (diversifiable). This distinction permits to apply two rules for risk-sharing: mutualisation of the former component, which is then completely covered, and securitization of the latter, which is shared with the insurer. Arrow (1996) underlines the fact that risk transfer contracts observed in real world mainly cover individual risks: each individual agent doesn't want to be risk-bearer and this function rests upon insurance companies. Participating contracts present another advantage: risk sharing doesn't necessary implies an intermediation with portfolio diversification in general and reinsurance in particular. The individuals can insure themselves with existing contracts.

Risk securitization is an alternative to insurance because it allows the insurer to transfer an excessive risk to financial markets. It is a useful tool that permits to break up a risk rather than to deal with its totality. Doherty and Schlesinger (2002) show that this distinction procures an increasing in the policyholders' welfare. Then, the idea is to substitute a financial contract to the "classical" non-participating contract in order to better hedge the systemic risk. This idea is developed by Mahul (2001 and 2002) who proves that the introduction of financial contracts is a market enhancing instrument.

Participating policies are nowadays used in cars and health insurance. In France, since 2003, the decrease of roads accidents due to the reinforcement of policy controls was quickly passed on the premia level of mutual benefit societies. In December 2006, a French insurer (Mutuelles du Mans Assurances) proposed a specific participating health insurance: with a 15% premium increase compared to standard contracts, the contribution is divided in two parts. The insurer collects the first one whereas the second one is put apart and operates as a reimbursable reserve. During the following year, if health expenses are low or null, then the insurer reimburse all or part of this reserve. By this way, the solidarity principle is corrected by the individual risk but, by construction, this kind of contract is made for low-risk individuals. In the agricultural sector, the states progressively decided to replace their global catastrophic coverage funds by an individual private and subsidized insurance. These contracts are generally restricted to catastrophic events, so their coverage subject to a basis risk is not incentive.

In this article, we refer to the models of Doherty and Schlesinger (2002) and Mahul (2002) who distinguish idiosyncratic and systemic risks and introduce participating and non-participating contracts. We first prove that under the classical assumption of fair insurance, their two approaches are equivalent. Then, the optimum design of insurance contracts is calculated with a generalization of Mahul (2002) relaxing the assumption of fair insurance. In particular, we introduce in our analysis financial indexes straightly correlated the systemic risk but subject to a basis risk. Then, we prove that the combination of both participating and financial contracts offers an unbiased coverage that eliminates the basis risk and provides a sustainable solution for the insurer and the stakeholder. This is done in the case of the development of specific crop risk insurance contracts.
2. The model

- General notations

The model is developed under expected utility theory. A risk-averse firm has an initial non-random welfare $w_0$ subject to a risk of loss $\ell \in \left[0, w_0\right]^1$. We assume this loss can be separated into two components: an individual one, $\bar{x}$ and a systemic one, $\bar{\varepsilon}$. Thus, we rewrite:

\[
(1) \quad \bar{\ell} = l(\bar{x}, \bar{\varepsilon})
\]

With $\bar{x} \geq 0$, $\bar{\varepsilon} \geq 0$, $l_x \geq 0$, $l_\varepsilon \geq 0$ and $E(\bar{\varepsilon}) = 0$. We also assume that $\bar{\ell}$, $\bar{x}$ and $\bar{\varepsilon}$ are commonly identified by everyone. Among the stakeholders, $\bar{x}_i$ are independent and are not correlated with the common risk $\bar{\varepsilon}$.

For example, let's consider a pool in which all the members are located in a floodable watershed. The $\bar{\ell}$ risk is then defined as the individual exposure to flood risk. The $\bar{\varepsilon}$ risk would be the common uncertainty that affects all the members of the pool, i.e. flood intensity. Over the years, $\bar{\varepsilon}$ is very often slightly negative as no flood event occurs but in some rare occasions, $\bar{\varepsilon}$ is widely positive. The $\bar{x}$ risk would be the impact of local parameters on the individual losses, which can be considered as independent among the members of the risk pool. The $\bar{x}$ risk seems to be partially diversifiable at the risk pool level provided the size of the insurer's portfolio is sufficiently large so that the law of great numbers applies. However, it's not the case for the $\bar{\varepsilon}$ risk unless this risk is spread with other groups exposed to different catastrophes or transferred to financial markets.

- Form of the loss function

The form of this diversification is straightly linked to the form of $l = l(x, \varepsilon)$. Consider the additive form: $l = l(x, \varepsilon) = x + \varepsilon$. In this case, we assume that losses can be decomposed onto two additive risks. For example, $\varepsilon = 10$ implies that all individual losses increase of 10 monetary units because of occurrence of a catastrophe. Then, Mahul (2001) shows the insurance coverage can be replicated by the combination of a "classical" non-participating insurance policy and a futures contract to cover the systemic component of the risk. In theory, the stakeholders can realize this combination by themselves, which is potentially interesting in order to reduce transaction costs using financial markets.

Let's consider now a multiplicative form for $\bar{\ell}$: $l = l(x, \varepsilon) = x(1 + \varepsilon)$. For example, $\varepsilon = 0.10$ implies that all individual losses increase by 10% because of the occurrence of a catastrophe. In this case, Doherty and Schlesinger (2002) indicate that the intervention of an insurer is necessary. Securitization comes from the pooling of the individual risks into the insurer's portfolio. The optimal design is then the combination a variable participating contract for the stakeholder and a futures contract for the insurer in order to cover its aggregate risk of dividends.

---

1 The ~ symbol indicates random variables whereas variables without tilde are realizations of random variables.
• **The contracts**

We consider that each individual or firm can construct a variable participating insurance policy by buying two kinds of contracts: a non-participating policy and a fully participating policy. Each policy is defined by a schedule, i.e. the premium and the indemnity.

By definition, the indemnity of the participating contract depends both on the idiosyncratic and systemic risks:

\[(2) \quad I(x, \varepsilon) \geq 0, \forall x, \varepsilon.\]

The premium is variable and depends on the occurrence of systemic risk. The individual risk is assumed to be insurable without any transaction cost and the insurer's portfolio is supposed to be large enough so that the law of great numbers and the mutualisation principle apply. The premium is then defined as the mathematical expectation of the systemic risk conditional indemnity, that is:

\[(3) \quad P(\varepsilon) = E(I(\bar{x}, \varepsilon)) = c(I(\bar{x}, \varepsilon))\]

The expectation is a function \(c\), which has the following properties: \(c(0) = 0\) and \(c'(\zeta) \geq 1, \forall \zeta\). Function \(c\) includes transaction costs in the calculus of the premium in addition to damage expected expenditures. The premium \(P\) is potentially subject to ex post adjustments. It can be fixed for catastrophes of mean size and revised at the end of the year to reflect the occurrence or the non-occurrence of an event. Thus, the systemic risk is filtered by this kind of contract, which is only able to insure the idiosyncratic component of the risk.

As in the former case, the indemnity of non-participating contract is written as a function of idiosyncratic and systemic risks:

\[(4) \quad J(x, \varepsilon) \geq 0, \forall x, \varepsilon.\]

The premium is fixed ex ante and defined by the mathematical expectation of the idiosyncratic and systemic risk conditional indemnity to which a loading factor applies, that is:

\[(5) \quad Q = (1 + \lambda) E(J(\bar{x}, \varepsilon)) = (1 + \lambda) d(J(\bar{x}, \varepsilon))\]

The expectation is a function \(d\), which has the following properties: \(d(0) = 0\) and \(d'(\zeta) \geq 1, \forall \zeta\). \(d\) includes transaction costs in the calculus of the premium in addition to damage expected expenditures. Under the assumption of risk-neutral insurers, the market price of the \(\varepsilon\) risk, as defined by Schlesinger (1999), is represented by the loading factor \(\lambda\).

Classically, the fixed premium of the non-participating contract is a tool to insure the systemic risk of a firm. This implies that the insurer supports the risk. To secure its contract and cover the \(\varepsilon\) risk, it is possible to buy financial contract based on catastrophe indexes.
• **Index and Basis Risk**

To link catastrophe events and economic losses, indexes have been computed in the major financial centres, such as CBOT in Chicago or LIFFE in London. 90% of weather derivatives have an underlying asset based on temperature. The most famous are the heating/cooling degree-days\(^2\), based on cumulative temperatures. There also exist derivatives based on rainfall but their market is still confidential. These index are assumed to be straightly related to climatic catastrophic losses so that the systemic component \(\varepsilon\) can be rewritten as a function of a loss index called \(\bar{z}\). This index is generally computed at a global scale so that it's not perfectly correlated to the individual systemic component of risk \(\varepsilon\). Thus, each financial contract written on \(\bar{z}\) is exposed to basis risk whose consequences must be examined.

Let's rewrite \(\varepsilon\) as:

\[
\varepsilon = \tilde{\varepsilon}(\bar{z},\tilde{b}) = \psi + \varphi \bar{z} + \tilde{b}, \text{with } \psi \geq 0, \varphi > 0, E(\tilde{b}) = 0
\]

The basis risk, \(\tilde{b}\), is assumed to be independent of the index \(\bar{z}\), the risk pool's systemic component \(\varepsilon\) and the idiosyncratic risk \(\tilde{x}\). For example, \(\tilde{x}\) denotes the aggregate loss of a regional pool and the \(\bar{z}\) index represents the national aggregate losses. Without loss of generality and to simplify the notations, we assume now that \(\psi = 0\) and \(\varphi = 1\).

Using the properties of \(\tilde{b}\) in (6), this leads to the following properties:

\[
(\tilde{6})' \quad \tilde{\varepsilon}(\bar{z},\tilde{b}) = \tilde{\varepsilon}(\bar{z}) \quad \text{and} \quad E(\tilde{\varepsilon}) = E(\bar{z})
\]

Then, we replace \(\varepsilon\) by \(z\) and the indemnity will depend on the catastrophe index, under the basis risk \(\tilde{b}\).

\[
(7) \quad J(x,\varepsilon) = J(x,\varepsilon(z)) = J(x,z) \geq 0, \forall x, z
\]

The premium remains fixed. The loading factor \(\lambda\) includes administrative costs and the insurer's risk aversion against the basis risk \(\tilde{b}\).

\[
(8) \quad Q = (1 + \lambda) E(J(\tilde{x},\tilde{z})) = (1 + \lambda) d(J(\tilde{x},\tilde{z})), \lambda \geq 0
\]

Finally, the combination of a participating and a non-participating insurance policy, called variable participating policy, is sold at price \([P(\varepsilon) + Q]\) and procures an indemnity equal to \([J(x,\varepsilon) + J(x,z)]\) when the effective values of the individual and systemic components and of the index are respectively \(x, \varepsilon\) et \(z\).

We can write final wealth as:

\(2\) A degree-day gauges the amount of heating or cooling needed for a building using 65 Fahrenheit degrees as a baseline. To compute heating/cooling degree-days, take the average temperature for a day and subtract the reference temperature of 65 Fahrenheit degrees. If the difference is positive, it is called a “Cooling Degree Days”. If the difference is negative, it is called a “Heating Degree Days”. The magnitude of the difference is the number of days and this information is utilized to calculate the individual needs.
The stakeholder has a twice-differentiable von Neumann-Morgenstern utility function $u(.)$, with $u' > 0$ and $u'' < 0$.

The problem of the risk-averse firm is to determine the optimal indemnity and the premium of both participating and non-participating policies that maximize the expected utility of its final wealth under the constraints (2), (3), (7) and (8):

$$\text{Max } Eu\left(w_0 - I(\tilde{x}, \tilde{e}) + I(\tilde{x}, \tilde{e}) - P(\tilde{e}) + J(\tilde{x}, \tilde{z}) - Q \right)$$

3. Generalization of previous results of the literature

The model we propose is very general and recovers classical problems about participating contracts. Proposition 1 demonstrates the possible equivalence between the model and Doherty and Schlesinger's (2002) approach.

**Proposition 1**: Under a multiplicative loss function $l(\tilde{x}, \tilde{e}) = (1 + \tilde{e})\tilde{x}$ and fair premia, resolving equation 10: $\text{Max } Eu\left(w_0 - I(\tilde{x}, \tilde{e}) + I(\tilde{x}, \tilde{e}) - P(\tilde{e}) + J(\tilde{x}, \tilde{z}) - Q \right)$ is equivalent to resolving the following problem: $\text{Max } Eu\left(w_0 - T(x, \epsilon) - (1 - \alpha)(1 + \epsilon)x \right)$, with $T(x, \epsilon) = \alpha E(x)[1 + \beta \lambda + (1 - \beta)\epsilon]$.

Where: $\alpha$ is the proportion of loss indemnified by the insurer and $\beta$ is a stakeholder choice variable denoting the degree of participation, with $\beta = 1$ denoting a fixed premium and $\beta = 0$ denoting full participation.

The proof is given in Appendix 1. Our model proposes to determine four optimal values whereas traditional approaches tend to determine proportions. This gives superiority to our model for the determination of the optimal premia and indemnities of the two separate participating and non-participating contracts. Determining $I, J, P$ and $Q$ yields $\alpha$ and $\beta$, whereas the reciprocal is not evident. Consequently, our model contributes to unify two parallel branches of the literature. Corollary 1 is an illustration of this possibility.

**Corollary 1 - Doherty and Schlesinger's (2002)**: Assuming a linear expectation function and $u'(\zeta) = 1, \forall \zeta$. Proposition 1 becomes: $\text{Max } w_0 - T(x, \epsilon) - (1 - \alpha)(1 + \epsilon)x$, where: $T(x, \epsilon) = \alpha E(x)[1 + \beta \lambda + (1 - \beta)\epsilon]$ is defined as the general premium of the variable participating contract.

The proof is given at the end of Appendix 1. With little adaptations, our formulation recovers Doherty and Schlesinger (2002). Consequently, our model maximizes the difference between initial wealth, the individual loss plus the result of the coverage (indemnities minus premia) while Doherty and Schlesinger's maximizes the difference between initial wealth and the premia paid plus non-covered losses.
4. Optimal insurance contracts design

Proposition 2 shows the design of optimal fully participating and non-participating contracts based on the use of financial markets.

**Proposition 2:** The optimal indemnity of fully participating and non-participating contracts $I^*$ and $J^*$, solutions to problem (10), take the form:

(i) If $c(\zeta) < (1 + \lambda)d(\zeta), \forall \zeta,$ then there exist $D_1 \geq 0$ and $D_2 \geq 0$ such that $J^*(x,z) = J^*(x,\varepsilon) = \max(P(\varepsilon) - D_2; 0)$ and $I^*(x,\varepsilon) = \max(l(x,\varepsilon) - D_1; 0)$.

(ii) If $c(\zeta) > (1 + \lambda)d(\zeta), \forall \zeta,$ then there exist $D_3 \geq 0$ such that $J^*(x,z) = J^*(x,\varepsilon) = \max(l(x,\varepsilon) - D_3; 0)$ and $I^*(x,\varepsilon) = 0$.

The proof is given in Appendix 2. The main figure to consider before pricing the different contracts is about the existence of the participating contract as the non-participating contract is systematically used to cover the systemic risk. We readapt Mahul (2002) introducing transaction costs and financial contracts instead of standard non-participating contracts.

In fact, the price of the participating contract may be lower to the one of the non-participating contract, and then would exist: that's point (i). It may occur if the administrative costs plus the risk premium of the non-participating policy are upper than the administrative costs of the participating policy. In fact, the price of the non-participating policy straightly depends on the capacity of the insurer to share the systemic risk, which justifies point (i). This can be done by securitization on the financial markets, taking into account a basis risk. By definition, this risk is not diversifiable and it is passed to the stakeholders through a large premium rate. Thus, the rationale is to buy a participating contract to cover above a deductible $D_1$ the idiosyncratic risk that filters the systemic risk. Then, the former is fully covered by a non-participating contract above a deductible $D_2$. The important thing to notice is that the indemnity of the non-participating contract depends on the premium of the participating contract.

In practice, the insurers prefer to anticipate the occurrence of a catastrophe and then they artificially increase the premium in order to avoid a possible ex-post default from the stakeholders. If no event occurred, then a dividend is distributed. This mechanism increases ex-ante the cost of subscribing a participating contract even if the probability to recover money is not negligible. For identical transaction costs, the risk premium $\lambda$ increases the cost of the participating contract, which leads to point (ii): this kind of contract is not subscribed and insurance is displayed only with the non-participating contract. This case is standard in the literature (and in practice). Then, the stakeholder is fully covered above a deductible $D_3$.

We only look at the case when participating policy is subscribed and Proposition 3 characterizes the optimal level of deductibility.

**Proposition 3:** Supposing that participating contracts exist, i.e. $c(\zeta) < (1 + \lambda)d(\zeta), \forall \zeta$:

(i) The optimal deductible $D_1$ of the participating policy equals zero if the premium is actuarially fair, i.e. $c'(\zeta) = 1, \forall \zeta,$ whereas it is positive if the premium is unfair, i.e. $c'(\zeta) > 1, \forall \zeta.$
The optimal deductible $D_2$ of the non-participating policy equals zero if the premium is actuarially fair, i.e. $\lambda = 0$ and $d'(\zeta) = 1, \forall \zeta$, whereas it is positive if the premium is unfair, i.e. $(1 + \lambda) d'(\zeta) > 1, \forall \zeta$.

The proof is given in Appendix 3. This result is conforming to standard insurance literature. With fair premia, the deductible and the loading factor are null and with unfair premia, at least one of them is strictly positive. One should notice that the double condition is necessary to obtain $D_2 = 0$.

Propositions 2 and 3 characterize the optimal insurance strategy with participating and non-participating contracts. Finally, both participating and non-participating policies can be combined to construct what is usually called a variable participating insurance contract, whose indemnity and premium are respectively:

\begin{align}
A(x, \varepsilon(z)) &= I(x, \varepsilon) + J(x, z) \\
B(\varepsilon) &= P(\varepsilon) + Q
\end{align}

This differs from Mahul (2002) because the indemnity of the non-participating policy depends on a financial index instead of pure systemic risk. Inserting the optimal values of $I, J, P$ and $Q$ in (11) and (12) gives:

\begin{align}
A(x, \varepsilon) &= \text{Max}[I(x, \varepsilon) - D_1; 0] + \text{Max}[P(\varepsilon) - D_2; 0] \\
B(\varepsilon) &= E\left( (\text{Max}[I(\tilde{x}, \varepsilon) - D_1; 0]) \right) + (1 + \lambda) d\left( E\left( \text{Max}[P(\tilde{\varepsilon}) - D_2; 0] \right) \right)
\end{align}

The financial index doesn't clearly appear in the former expressions because it is hidden by the $\varepsilon$ value. The optimal strategy of coverage with a variable participating contract permits to see the interest of such formulation.

5. Optimal strategy of coverage

- Back to real market, introducing index-based securities

In practice, two types of contracts are sold on real markets, participating one and non-participating one exclusively based on a financial index $z$ straightly correlated to the pure systemic risk $\varepsilon$. Thus, the indemnity of the financial contract is written as follows:

\begin{align}
K(z) &\geq 0, \forall z \\
\tilde{\varepsilon} &\equiv \varepsilon(z) = \tilde{z} + \tilde{b}, E(\tilde{b}) = 0
\end{align}

The premium remains fixed and the loading factor $\lambda$ still includes transaction and administrative costs and the insurer's risk aversion against the basis risk $\tilde{b}$.
(16) \( Q = (1 + \lambda) E \left( K(\tilde{z}) \right) \)

In the former expression, the lack of the idiosyncratic risk in the equations (15) and (16) is justified by its coverage using a participating contract.

Using Proposition 2 and 3 for the participating policy, the indemnity and the premium of contracts sold on real markets become:

(17) \( I^*(x, \varepsilon) = \text{Max} \left( I(x, \varepsilon) - D_1; 0 \right) \)

(18) \( P(\varepsilon) = c \left[ E \left( \text{Max} \left( I(\tilde{x}, \varepsilon) - D_1; 0 \right) \right) \right] \)

With: \( D_1 = 0 \) if \( c'(\cdot) \equiv 1 \) and \( D_1 > 0 \) otherwise.

For the non-participating policy, we similarly obtain:

(19) \( K^*(z) = \text{Max} \left( P(\varepsilon) - D_2; 0 \right) \)

(20) \( Q = (1 + \lambda) E \left( \text{Max} \left( P(\tilde{\varepsilon}) - D_2; 0 \right) \right) \)

With: \( D_2 = 0 \) if \( \lambda = 0 \) and \( D_2 > 0 \) otherwise.

The aim is now to examine the optimal strategy of coverage, using first a participating contract, second a non-participating contract, and third the combination of both types of contracts.

- **Participating contract**

First selecting the additive form of the loss function, i.e. \( I(x, \varepsilon) = x + \varepsilon \), equations (17) and (18) relative to the participating contract become:

(21) \( I^*(x, \varepsilon) = x + \varepsilon \)

(22) \( P(\varepsilon) = E(\tilde{x}) + \varepsilon \)

The stakeholder's final loss is then equal to yield loss plus the difference between the premium and the indemnity of the participating contract:

\[
\text{Loss}^*_p = I(x, \varepsilon) + P(\varepsilon) - I(x, \varepsilon) = x + \varepsilon + P(\varepsilon) - \text{Max} \left[ x + \varepsilon - D_1; 0 \right]
\]

\[
= \begin{cases} 
 x + \varepsilon + P(\varepsilon) & \text{if } x \leq D_1 - \varepsilon \\
 D_1 + P(\varepsilon) & \text{if } x \geq D_1 - \varepsilon 
\end{cases}
\]

Similarly, selecting the multiplicative form of the loss-function, i.e. \( I(x, \varepsilon) = x(1 + \varepsilon) \), equations (17) and (18) become:

(24) \( I^*(x, \varepsilon) = x(1 + \varepsilon) \)
The stakeholder's final loss is then equal to yield loss plus the difference between the premium and the indemnity of the participating contract:

\[
\text{Loss}_{PC}^+ = l(x, \varepsilon) + P(\varepsilon) - I(x, \varepsilon) = x(1 + \varepsilon) + P(\varepsilon) - \text{Max}[x(1 + \varepsilon) - D_i; 0]
\]

Under the subscription of a participating contract, the stakeholder's loss always depends on the systemic component \(\varepsilon\) but for large idiosyncratic losses, i.e. \(x \geq D_i - \varepsilon\) for additive losses and \(x \geq D_i / (1 + \varepsilon)\) for multiplicative losses, we observe that it only depends on \(\varepsilon\). Thus, the participating contract offers a perfect coverage against the idiosyncratic risk but it completely filters the systemic risk, which is not covered at all.

**Proposition 4:** Eliciting \(P(\varepsilon)\) under the assumption that the contract is sold at a fair price, i.e. \(c'(\zeta) = 1, \forall \zeta\) and consequently \(D_i = 0\) (Proposition 3-i), we get:

\[
\begin{align*}
(i) & \quad \text{Loss}_{PC}^+ = E(\tilde{x}) + \varepsilon \\
(ii) & \quad \text{Loss}_{PC}^x = E(\tilde{x})(1 + \varepsilon)
\end{align*}
\]

Proof is given in Appendix 4. Under usual assumptions in the literature, the former expression clearly indicates that the farmer is fully protected against its individual risk whereas the systemic risk is not insured, which is damageable for the stakeholders.

Therefore, in our approach, the non-participating coverage is only used to protect against the systemic risk. We examine now the optimal strategy of coverage against the systemic risk using a financial non-participating contract.

- **Financial non-participating contract**

The interest is now to focus on the replication of the financial non-participating contract, which is given by Proposition 5.

**Proposition 5:** By definition of \(z\) in (6) and of \(K^*(z)\) in (19), the optimal strategy of replication of the non-participating contract is given by:

\[
\begin{align*}
(i) & \quad K^+(z) = \text{Max}[z + E(\tilde{x}) - D_2 + b; 0], \text{ for an additive loss function.} \\
(ii) & \quad K^x(z) = E(\tilde{x}) \text{Max}[z + 1 - D_2 / E(\tilde{x}) + b; 0], \text{ for a multiplicative loss function.}
\end{align*}
\]

Proof is given in Appendix 5. The first equation means that the optimal non-participating contract can be replicated by purchasing a call option at a strike price equal to \(E(\tilde{x}) - D_2\), subject to the basis risk \(b\). The second equation means that the optimal non-participating
contract can be replicated by purchasing a number of $E(\tilde{x})$ call options at a strike price equal to $1 - D_2 / E(\tilde{x})$, subject to the basis risk $b$.

**Proposition 6:** Using the properties of $z$ given by (6'), the loss after the subscription of the financial non-participating contract is equal to:

(i) $\text{Loss}^+_{\text{NPC}} = Q - K^+(z) = [E(\tilde{z}) - z - b] + \lambda E[K^+(z)]$, for an additive loss function.

(ii) $\text{Loss}^x_{\text{NPC}} = Q - K^-(z) = E(\tilde{x})[E(\tilde{z}) - z - b] + \lambda E[K^-(z)]$, for a multiplicative loss function.

Proof is given in Appendix 6. With additive losses, the optimal strategy is to sell an unbiased futures contract on $z$, subject to the basis risk $b$ whereas the optimal strategy with multiplicative losses is to sell a number of $E(\tilde{x})$ unbiased futures contracts on $z$, subject to the basis risk $b$. However, in each case, losses are increased by the loading ratio multiplying the expectation of the indemnity $K(z)$. This corresponds to the lack of indemnification associated to the supplementary premium of the non-participating contract.

**Corollary 6:** Assuming the financial contract is sold at a fair price, i.e. $D_2 = 0$ and $\lambda = 0$, the loss after the subscription of the financial non-participating contract is equal to:

(i) $\text{Loss}^+_{\text{NPC}} = Q - K^+(z) = E(\tilde{z}) - z - b$, for an additive loss function.

(ii) $\text{Loss}^x_{\text{NPC}} = Q - K^+(z) = E(\tilde{x})[E(\tilde{z}) - z - b]$, for a multiplicative loss function.

Proof is trivial. We obtain a quite standard result in the literature (Mahul, 2002), i.e. an unbiased coverage with futures contracts only subject to a basis risk. Differently said, the efficiency of the coverage depends on the correlation between $E(\tilde{z})$ and $\varepsilon$. Moreover, the basis risk is proportional to the expectation of the idiosyncratic risk.

- **Variable participating contract**

As defined before, the variable participating contract is the combination of a participating contract and a non-participating contract. The strength of such a strategy is to get a more efficient coverage, as shown by Proposition and Corollary 7.

**Proposition 7:** Using the properties of $z$ given by (6'), total loss after the subscription of the variable non-participating contract is equal to:

(i) $\text{Loss}^+_{\text{NC, NPC}} = E(\tilde{x}) + E(\tilde{z}) + \lambda E[K^+(z)]$, for an additive loss function.

(ii) $\text{Loss}^x_{\text{NC, NPC}} = E(\tilde{x})[1 + E(\tilde{z})] + \lambda E[K^+(z)]$, for a multiplicative loss function.

Proof is given in Appendix 7. This result provides two major advantages of our combination. First, the variable participating contract neutralizes the basis risk generated by the use of financial products. Second, both idiosyncratic and systemic risks are covered and the initial loss is transformed in expectations of the idiosyncratic and financial components. In counterpart, the systemic risk is replaced by a financial risk and there still remains transaction costs.
Corollary 7: Assuming the financial contract is sold at a fair price, i.e. $D_2 = 0$ and $\lambda = 0$, total loss after the subscription of the variable participating contract is equal to:

(i) $\text{Loss}^+_{\text{VPC}} = E(\bar{x}) + E(\bar{z})$, for an additive loss function.

(ii) $\text{Loss}^x_{\text{VPC}} = E(\bar{x})[1 + E(\bar{z})]$, for a multiplicative loss function.

Proof is trivial. Referring to classical assumption adopted in the literature, this combination of the participating and the financial (non-participating) contracts creates a perfect unbiased coverage. In particular, one should notice there is no covariance term associated to a multiplicative loss function. This gives an argumentation in favour of the subscription of both participating and non-participating contracts by exposed stakeholders. Index-based securities exist and are frequently the only one subscribed despite the basis risk and the incomplete coverage of the idiosyncratic risk. Proposition 7 affirms the theoretical interest to use participating contracts in complement of index-based non-participating contracts. Assuming fair premia, the standard result is:

$$\begin{align*}
(27) & \quad \left\{ \begin{array}{l}
I^+(x, \varepsilon) = x + \varepsilon = x + z + b \rightarrow E(\bar{x}) + E(\bar{z}) \\
I^x(x, \varepsilon) = x(1 + \varepsilon) = x(1 + z + b) \rightarrow E(\bar{x})(1 + E(\bar{z}))
\end{array} \right.
\end{align*}$$

Initial losses are transformed in an interesting way for the stakeholder, providing transaction costs are eliminated. The aim is now to study how this interesting result can be applied to crop insurance contracts.

6. Implications for crop insurance contracts

In developed countries, crop insurance contracts are more and more proposed to the farmers in substitution to global and/or emergency indemnification fund.

Such systems exist worldwide and present advantages for the different actors:

- Historically, the states used to manage a catastrophe fund. Such a way implies to create a catastrophe fund in order to face the different catastrophe. In theory, the states guarantee is unlimited but for budgetary constraints, the indemnifications are restricted in practice. By encouraging (and controlling) private insurance, e.g. France subsidize the premia of private crop insurance contracts, the states limit their implication and try to improve efficiency in the coverage of catastrophe events.

- Encouraged by a favourable legislative environment, private or mutual insurers tend to offer more and more catastrophic coverage. The potential market is considerable and they have an intermediation role to play. The development of financial markets dedicated to catastrophes permits to absorb an increasing part of the systemic risk. Nevertheless, transaction and administrative costs limit the efficiency of the different contracts.

- For the reasons explained before, the farmers were imperfectly covered by public insurance. Insurance and financial contracts offer the possibility to individualize the protection. In fact, the underlying indexes under which crop contracts are elaborated are global and the stakeholder is subject to an important basis risk. Added to unfair premia, this explains why such contracts are not subscribed if they are not subsidized.

Faced to structural deficits of their catastrophic funds, the USA reformed their system in 1996 with the "Fair Act" introducing an improperly named "revenue insurance", which is in reality a crop revenue insurance. The insurers propose different contracts whose premia are 60%
subsidized so that 70% of the agricultural surfaces are now covered against climatic risks. France decided to adopt a similar system in 2005\(^3\) and extended it in 2006\(^4\) encouraging private insurance. Public subvention is fixed equal to 35% of the premia and it is coupled with a deductible equal to 25% of insured capital. A ministry report confirms 60% of the agricultural surfaces are now covered with crop insurance.

However, such systems face three main difficulties:

- First, the number of policyholders is not optimal because one third of the farms are not protected. It is damageable for risk mutualisation and the premia level.
- Second, the states intervention is necessary to guarantee the solvency of this new kind of insurance.
- Third, there is the problem of reinsurance because it is almost known that the global market of agricultural insurance cannot face the amount of damages of a catastrophic year.

Our theoretical framework provides answers to these three majors limits of crop insurance systems. In particular, we can prove how the combination of participating and financial contracts can encourage the development of such insurance using financial markets.

For crops, we assume that the losses take a multiplicative form, i.e. \( l(x, \varepsilon) = x(1+\varepsilon) \), with our usual notations. This realistic hypothesis means that losses increase by \( \varepsilon \% \) due to the occurrence of a catastrophe\(^5\). Applied to agricultural crop insurance, \( \varepsilon \) is yield shortfall caused by local weather events and \( x \) is long-term average individual crop loss based on crop price at harvest. \( \varepsilon \) and \( x \) are supposed independent.

Facing crop yield risks, the farmer can subscribe both participating and non-participating contracts. As proved with Proposition 2, he will select first the participating contract to cover its sole idiosyncratic risk. Therefore, in our approach, the non-participating coverage is only used to protect against the systemic risk. We assume there exist an individual crop yield index\(^6\) noted \( z \) that is straightly correlated to the systemic risk, as defined by equation (6). For example, this index can be based upon cumulated degree-days to cover yield shortfall after harvest, monthly precipitations to cover drought and daily precipitations to cover floods. In counterpart, there exist a basis risk \( b \) due to the imperfect linkage between the index and the reality. Subscribed apart, this contract (similar to existing one) can already be subscribed but it is clearly inefficient and even more if transaction costs and risk premia remain.

However, we proved that associated to a participating policy, the coverage is more efficient because of the neutralization of the basis risk. Then, this is more inciting for farmers to cover

---


\(^5\) The reasoning that will follow can strictly be applied to additive losses with similar results as proved by Proposition 6.

\(^6\) Consequently, each farmer constructs its own multi-crop insurance by subscribing the different financial contracts corresponding to its cultures. It is typically the case in the U.S. system. The insurers can also propose a portfolio, which would be representative of the different types of cultures of a farm.
their losses. It is also an encouragement for insurers to propose crop insurance contracts based on financial products, which transfers the systemic risk to financial markets and contributes to resolve the reinsurance limitations. To procure incitement for the farmers to subscribe variable participating contracts, the role of the state should be double: first, subsidize the financial contract of an amount equal to the risk premia loss so that the financial contract would be fairly tariffed for farmers; second, encourage the subscription of an additional participating policy, in order to obtain the advantages of Corollary 7, i.e. a perfect unbiased coverage.

7. Conclusion

Variable participating contracts present several advantages that should justify their development. They are interesting for all actors: the states limit their intervention to the subsidization of the premia of both participating and financial contracts. Insurers minimize their potential losses by simply covering the idiosyncratic risk through a variable premium and securitizing the idiosyncratic risk on financial markets. Finally, stakeholders take full benefits from the combination of the two contracts, which annihilates the basis risk linked to the financial index. In the general case, with unfair insurance, the risk premium affects the stockholder's losses with an additive element. Then we can suppose the states will take into account this fact and subsidize the premium so that the insurance becomes fair.

Applied to crop insurance, the perspectives are promising as more and more countries decide to reform their agricultural coverage against climatic events. The variable participating policies are a credible way to increase the number of policyholders and enhance this recent market. Further research should investigate practical approaches and should test the potential subscription of variable participating contracts by farmers in replacement of actual policies.

References


Appendix 1 – Proof of Proposition 1

We start from our original problem:

(A1) \[
\max_{I,J,P,Q} Eu \left( w_0 - I(x,\varepsilon) + J(x,\varepsilon) - P(\tilde{x},\tilde{\varepsilon}) + J(\tilde{x},\tilde{\varepsilon}) - Q \right).
\]

The equivalence is obtained by simplifications and the elicitation of \(\alpha\), the proportion of loss indemnified by the insurer and \(\beta\) denoting the degree of participation. \(\alpha\) is defined by the following equality:

(A2) \[ I(\tilde{x},\tilde{\varepsilon}) + J(\tilde{x},\tilde{\varepsilon}) = \alpha \times l(\tilde{x},\tilde{\varepsilon}) \]

\(\beta\) is the degree of participation. It can be elicited when replacing \(P\) and \(Q\) by their value when \(c\) and \(d\) functions are linear, with \(c'(\zeta) = 1\) and \(d'(\zeta) = 1, \forall \zeta\). We also consider a multiplicative loss function:

(A3) \[ l(\tilde{x},\tilde{\varepsilon}) = (1 + \tilde{\varepsilon})\tilde{x} \]

Under our notations, it comes:

(A4) \[ P(\tilde{\varepsilon}) = \alpha (1 - \beta) E[I(x,\varepsilon)] = \alpha (1 - \beta) E(x)(1 + \varepsilon) \]

(A5) \[ Q = \alpha \beta (1 + \lambda) E(x) \]

Replacing (A2), (A3), (A4) and (A5) in (A1) permits to obtain the following maximization:

(A6) \[
\max_{\alpha,\beta} Eu \left( w_0 - T(x,\varepsilon) - (1 - \alpha)(1 + \varepsilon)x \right), \text{ with } T(x,\varepsilon) = \alpha E(x) \left[ 1 + \beta \lambda + (1 - \beta) \varepsilon \right].
\]

Where \(\alpha\) and \(\beta\) are now to be determined instead of \(I, J, P\) and \(Q\).

Proof of Corollary 1

Assuming the expectation function is linear and \(u'(\zeta) = 1, \forall \zeta\), (A6) becomes:

(A7) \[
\max_{\alpha,\beta} w_0 - T(x,\varepsilon) - (1 - \alpha)(1 + \varepsilon)x
\]
We consider then \( T(x, \varepsilon) = \alpha E(x) \left[ 1 + \beta \lambda + (1 - \beta) \varepsilon \right] \) as the global premium of the variable participating contract.

This is Doherty and Schlesinger's (2002) model.

**Appendix 2 – Proof of Proposition 2**

Problem (10) can be solved using Karush-Kuhn-Tucker conditions for \( I(x, \varepsilon) \) and \( J(x, z) \) because their first derivatives appear neither in the objective function nor in the constraints.

\[
\begin{align*}
\text{Max } & \sum_{i,j,p,q} E u \left( w_0 - l(x, \varepsilon) + I(x, \varepsilon) - P(\varepsilon) + J(x, z) - Q \right) \\
\text{subject to:} & \\
& I(x, \varepsilon) \geq 0 \text{ associated with } \lambda_1(x, \varepsilon) \\
& P(\varepsilon) = c \left( I(x, \varepsilon) \right) \text{ associated with } \mu_1(\varepsilon) \\
& J(x, z) \geq 0 \text{ associated with } \lambda_2(x, z) \\
& Q = (1 + \lambda) d \left( I(x, \varepsilon) \right) \text{ associated with } \mu_2
\end{align*}
\]

Where: \( \lambda_1, \lambda_2, \mu_1 \) and \( \mu_2 \) are Lagrange multipliers.

The first-order condition associated to the indemnity of the participating contract is:

\[
\frac{\partial L}{\partial I(x, \varepsilon)} = u' \left( w_0 - l(x, \varepsilon) + I(x, \varepsilon) - P(\varepsilon) + J(x, z) - Q \right) + \lambda_1(x, \varepsilon) - \mu_1(\varepsilon) c' \left( EI(x, \varepsilon) \right) = 0
\]

A supplementary condition for the maximisation is associated to the Lagrange multiplier of the indemnity of the participating contract:

\[
\lambda_1(x, \varepsilon) = \begin{cases} 
0 & \text{if } I(x, \varepsilon) > 0 \\
\geq 0 & \text{otherwise}
\end{cases}
\]

Considering a positive indemnification, (B2) can be rewritten as:

\[
\frac{\partial L}{\partial I(x, \varepsilon)} = u' \left( w_0 - l(x, \varepsilon) + I(x, \varepsilon) - P(\varepsilon) + J(x, z) - Q \right) = \mu_1(\varepsilon) c' \left( EI(x, \varepsilon) \right)
\]

At the optimum, the first derivative of the utilitarian function is supposed to be constant for each level of systemic (or financial) risk, remembering that participating contracts filters this kind of risk. Thus, for given \( w_0 \) and \( Q \), it comes that:

\[
I(x, \varepsilon) + J(x, z) = I(x, \varepsilon) + P(\varepsilon), \forall \varepsilon : I(x, \varepsilon) > 0
\]

We use the same reasoning for non-participating contracts. The first-order condition associated to the indemnity of the non-participating contract is:
A supplementary condition for the maximisation is associated to the Lagrange multiplier of the indemnity of the non-participating contract:

\[
\lambda_2(x,z) \begin{cases} = 0 & \text{if } J(x,z) > 0 \\ \geq 0 & \text{otherwise} \end{cases}
\]

Considering a positive indemnification, (B6) can be rewritten as:

\[
u'(w_0 - l(x,\varepsilon) + I(x,\varepsilon) - P(\varepsilon) + J(x,z) - Q) + \lambda_2(x,z) - \mu_2(1+\lambda)d'(EJ(\tilde{x},\tilde{z})) = 0
\]

At the optimum, the first derivative of the utilitarian function is supposed to be constant for each level of systemic (or financial) risk for each state of the world where \( J \) is paid, considering the non-participating contract protects against both idiosyncratic and systemic risks. Therefore, for given \( w_0 \) and \( Q \), it comes that:

\[
I(x,\varepsilon) + J(x,z) = l(x,\varepsilon) + P(\varepsilon), \forall (x,z): J(x,z) > 0
\]

Then, we must consider the stakeholder’s choice. The first subject is whether they include in heir insurance policy participating contracts. In fact, there the “classical” non-participating contract is always selected, as it is the only one that covers systemic risk. The second subject is about the form of the contract. Following Raviv (1979), when two risks \( x \) and \( \varepsilon \) (as defined in our paper) are insurable, then the insurance policy with a deductible on the aggregate losses is optimal.

For the subscription of the participating contract, two cases exist:

- The premium of the non-participating contract is higher than for the participating contract:

\[
Q > P(\varepsilon) \iff (1+\lambda)d'(\zeta) > c(\zeta), \forall \zeta
\]

Then, to cover the idiosyncratic risk \( x \), the cheapest contract is selected, i.e. the participating one, and the premium is defined taking into account total loss minus a deductible \( D_1 \), as follows:

\[
I^*(x,\varepsilon) = \text{Max}\{I(x,\varepsilon) - D_1; 0\}
\]

With respect to (B5), the premium of the non-participating contract depends on the variable premium of the participating contract minus a deductible \( D_2 \):

\[
J^*(x,z) \equiv J^*(x,\varepsilon) = \text{Max}\{P(\varepsilon) - D_2; 0\}
\]
The premium of the participating contract is higher than for the non-participating contract:

(B13) \( P(\varepsilon) > Q \Leftrightarrow c(\zeta) > (1 + \lambda) d(\zeta), \forall \zeta \)

Then, to cover the idiosyncratic risk \( x \), the cheapest contract is selected, i.e. the non-participating one. This implies that the participating contract is neither chosen:

(B14) \( I^*(x, \varepsilon) = 0 \)

Its premium is neither calculated and full insurance is only provided by the non-participating contract above a deductible \( D_3 \):

(B15) \( J^*(x, z) = I^*(x, \varepsilon) = \text{Max}\{I(x, \varepsilon) - D_3; 0\} \)

This is the standard result in literature when only non-participating contracts exist.

Appendix 3 – Proof of Proposition 3

The optimisation (B1) is now operated on \( Q \) and \( P(\varepsilon) \) to find the optimal level of deductibles \( D_1 \) and \( D_2 \). For practical purposes, we define:

(C1) \( \psi(x, \varepsilon) = w_0 - I(x, \varepsilon) + I(x, \varepsilon) - P(\varepsilon) + J(x, z) - Q \)

- For the non-participating contract, the first-order condition is:

(C2) \( \frac{\partial L}{\partial Q} = Eu'(\psi(\tilde{x}, \tilde{\varepsilon})) - \mu_2 = 0 \)

Replacing the value of \( \mu_2 \) in (B6) gives the following equality:

(C3) \( \lambda_2(x, z) = -u'(\psi(x, \varepsilon)) + Eu'(\psi(\tilde{x}, \tilde{\varepsilon}))(1 + \lambda) d'(EJ(\tilde{x}, \tilde{z})) \)

Taking the expectation of \( \lambda_2 \) yields:

(C4) \( E\lambda_2(\tilde{x}, \tilde{z}) = Eu'(\psi(\tilde{x}, \tilde{\varepsilon}))[1 + (1 + \lambda) d'(EJ(\tilde{x}, \tilde{z})) - 1] \)

Consequently, \( (1 + \lambda) d'(\zeta) = 1, \forall \zeta \) implies first that \( \lambda = 0 \) because \( d'(\zeta) \geq 1, \forall \zeta \) and second that \( E\lambda_2(\tilde{x}, \tilde{z}) = 0 \). Then, \( \lambda_2(x, z) = 0, \forall (x, z) \) because \( \lambda_2(x, z) \geq 0 \). Using (B7), it means that \( J(x, z) > 0 \). Thus, \( D_2 = 0 \). Similarly, \( (1 + \lambda) d'(\zeta) > 1, \forall \zeta \) implies \( D_2 > 0 \).

- For the participating contract, the optimisation is quite different because the premium is variable and depends on \( \varepsilon \). Thus, it is not possible to compute the first-order condition of problem (B1) by deriving the Lagrange function.

The solution is to replace \( I(\tilde{x}, \tilde{\varepsilon}) \) by its value found in Proposition 2:
Problem (B1) becomes:

\[ \max_{D_1} E u'(w_0 - l(x, \epsilon) + \max \{ l(x, \epsilon) - D_1; 0 \}) - P(\epsilon) + J(x, \tilde{z}) - Q \]

The first-order condition of this problem is:

\[ \frac{\partial L}{\partial D_1} = E u'(\psi(x, \epsilon)) + \lambda_i(x, \epsilon) - \mu_i(\epsilon) c'(\max \{ l(x, \epsilon) - D_1; 0 \}) = 0 \]

Replacing the value of \( \mu_i \) in (B2) and rearranging the expectation operator gives the following equality:

\[ E \lambda_i(x, \epsilon) = E u'(\psi(x, \epsilon)) \left[ c'(\max \{ l(x, \epsilon) - D_1; 0 \}) - 1 \right] \]

Without loss of generality, let's consider \( D_1 = 0 \), (C8) becomes:

\[ E \lambda_i(x, \epsilon) = E u'(\psi(x, \epsilon)) \left[ c'(l(x, \tilde{\epsilon})) - 1 \right] \]

Consequently, \( c'(\zeta) = 1, \forall \zeta \) implies \( E \lambda_i(x, \epsilon) = 0 \) and \( \lambda_i(x, \epsilon) = 0, \forall (x, \epsilon) \) because \( \lambda_i(x, \epsilon) \geq 0 \). Using (B3), it means that \( I(x, \epsilon) > 0 \). Then, \( D_1 = 0 \). Similarly, \( c'(\zeta) > 1, \forall \zeta \) implies \( D_1 > 0 \).

Appendix 4 – Proof of Proposition 4

Eliciting \( P(\epsilon) \) under the assumption that the contract is sold at a fair price, i.e. \( c'(\zeta) = 1, \forall \zeta \) and consequently \( D_1 = 0 \) (Proposition 3-i) gives:

\[ (D1) \quad \text{Loss}_{rc}^* = l(x, \epsilon) + P(\epsilon) - I(x, \epsilon) = x + \epsilon + E(\max \{ x + \epsilon; 0 \}) - \max \{ x + \epsilon; 0 \} = E(x) + \epsilon \]

\[ (D2) \quad \text{Loss}_{rc}^{-} = l(x, \epsilon) + P(\epsilon) - I(x, \epsilon) = x(1 + \epsilon) + E(\max \{ x(1 + \epsilon); 0 \}) - \max \{ x(1 + \epsilon); 0 \} = E(x)(1 + \epsilon) \]

This leads to points (i) and (ii).

Appendix 5 – Proof of Proposition 5

Supposing first an additive loss function, i.e. \( l(x, \epsilon) = x + \epsilon \), then, by definition of \( z \) in (6), of \( K^*(z) \) in (19) and \( P(\epsilon) \) in (22), the optimal strategy of replication of the non-participating contract is given by:
This leads to point (i).

Supposing now a multiplicative loss function, i.e. $l(x, \varepsilon) = x(1 + \varepsilon)$, then, by definition of $z$ in (6), of $K^*(z)$ in (19) and $P(\varepsilon)$ in (24), the optimal strategy of replication of the non-participating contract is given by:

(E2)
$$K^*(z) = \max [(1+\varepsilon)E(\tilde{x}) - D_2; 0] = E(\tilde{x})\max [\varepsilon + 1 - D_2 / E(\tilde{x}); 0] = E(\tilde{x})\max [z + 1 - D_2 / E(\tilde{x}) + b; 0]$$

This leads to point (ii).

Appendix 6 – Proof of Proposition 6

Let's consider the additive case with $l(x, \varepsilon) = x + \varepsilon$. By definition of $K^*(z)$ in Proposition 5, $Q$ in (20), $P(\varepsilon)$ in (22) and using the definition of $z$ in (6) and (6'), we obtain:

(F1)
$$Loss^*_{spc} = Q - K^*(z) = \max [(1 + \lambda)E(z + E(\tilde{x}) - D_2 + b) - z + E(\tilde{x}) - D_2 + b]$$
$$= [E(\tilde{z}) + E(\tilde{x}) - D_2 - z - E(\tilde{x}) + D_2 - b] + \lambda [E(z) + E(\tilde{x}) - D_2]$$
$$= [E(\tilde{z}) - z - b] + \lambda [E(\tilde{x}) + E(\tilde{x}) - D_2]$$
$$= [E(\tilde{z}) - z - b] + \lambda [K^*(z)]$$

This is point (i).

Let's consider now the multiplicative case with $l(x, \varepsilon) = x(1 + \varepsilon)$. By definition of $K^*(z)$ in Proposition 5, $Q$ in (20), $P(\varepsilon)$ in (25) and using the definition of $z$ in (6) and (6'), we obtain:

(F2)
$$Loss^*_{spc} = Q - K^*(z) = \max [(1 + \lambda)E(\tilde{x})E(z + 1 - D_2 / E(\tilde{x}) + b) - z + 1 - D_2 / E(\tilde{x}) + b]$$
$$= E(\tilde{x})\{[E(\tilde{z}) + 1 - D_2 / E(\tilde{x}) - z - 1 + D_2 / E(\tilde{x}) - b] + \lambda [E(z) + 1 - D_2 / E(\tilde{x})]\}$$
$$= E(\tilde{x})\{[E(\tilde{z}) - z - b] + \lambda [E(z) + 1 - D_2 / E(\tilde{x})]\}$$
$$= E(\tilde{x})[E(\tilde{z}) - z - b] + \lambda E[K^*(z)]$$

This is point (ii).

Appendix 7 – Proof of Proposition 7

Defining the variable participating contract as the simple combination of a participating and a non-participating contract, total loss is equal in case to the sum of the losses of the two contracts. For additive losses, it is the sum of (D1) and (F1), i.e.:
(G1) \[ \text{Loss}^+_{\text{NC-NPC}} = E(\tilde{x}) + \varepsilon + \left[ E(\tilde{z}) - z - b \right] + \lambda \left[ K^+(z) \right] = E(\tilde{x}) + E(\tilde{z}) + \lambda \left[ K^+(z) \right] \]

This leads to point (i).

For multiplicative losses, total loss is equal to the sum of (D2) and (F2), i.e.:

(G2)

\[ \text{Loss}^*_{\text{NC-NPC}} = E(\tilde{x})[1 + \varepsilon] + E(\tilde{x})\left[ E(\tilde{z}) - z - b \right] + \lambda E\left[ K^*(z) \right] = E(\tilde{x})[1 + E(\tilde{z})] + \lambda E\left[ K^*(z) \right] \]

This leads to point (ii).